7: Compounding Frequency

The factors developed in the preceding chapters all use the interest rate per compounding period as a parameter. This chapter examines the interest rate itself more closely, focusing on what happens as the frequency of compounding changes. For example, the debt owed on $100 after year 1 at 18% per year compounded annually is $118.00, but the debt is $119.56 at "18% per year compounded monthly."

This chapter shows how to convert common financial language into terms suitable for use with factors and how to compute the interest rate on a loan from the borrower's perspective. Then it explains how to analyze cash flows that occur at intervals other than the stated compounding period, and examines cash flows that occur continuously. All of these situations commonly occur in banking, industrial, and governmental analyses.

7.1 Nominal and Effective Interest

This section explains common financial terminology, and then it presents formulas that convert this terminology into numbers usable in financial computations.

Terminology

An interest rate of "18% per year compounded monthly" translates to "1½% per month" for computational purposes. If $100 is borrowed today, then the debt after one year is:

$$\$119.56 = 100\,(F|P,\ 1½\%,\ 12\text{-}0) \qquad (7\text{-}1)$$

The compounding period is one month, so time must be measured accordingly. For example, the last parameter in equation (7-1) is in terms of months (12-0) instead of years. The *only* use of the quoted 18% yearly rate is to compute the monthly rate.

An interest rate measures the speed at which debt or capital grows during a stated period of time. For example, if a debt of $100 would grow to $119.56 after a year, then its yearly rate of growth is 19.56%:

$$19.56\% = (119.56 - 100)\ /\ 100 \qquad (7\text{-}2)$$

The rate at which interest actually accumulates over a period is known as the *effective interest rate* or the *yield* for that period, so in this case the effective rate or yield is 19.56% per year. Even if the loan were repaid with $101.50 after 1 month, the effective rate *per year* is still 19.56%, the percentage by which the debt would increase if unpaid for 12 months. This is similar to someone running a mile in 4 minutes; the person does not have to run for an hour to compute the average speed of 15 miles per hour.

The 18% value can be misleading to someone who does not understand financial terminology and compounding. It is known as a *nominal* rate, meaning "in name only." Lenders tend to quote a nominal rate because it is less than the effective rate and hence more attractive. Conversely, institutions providing investment services tend to quote an effective rate or yield since it is greater than the nominal rate and best presents their offerings.

Another common rate is the *annual percentage rate* or *APR* that lenders disclose due to the federal Truth in Lending law. The APR formula is easy for someone without an education in economic analyses to use, but it only approximates the correct effective rate computed using the procedures described below.

Formulas

Let r be a nominal rate over a long period composed of P short compounding periods, such as 18% per year compounded monthly where the long period (one year) has 12 short compounding periods (months). Then the rate per compounding period is given by

$$\boxed{i_P = r / P} \,. \qquad (7\text{-}3)$$

Thus 18% per year compounded monthly implies a rate of 18% / 12 or 1½% per month. Compounding at rate i_P occurs P times during the long period, so an initial amount X would grow to $X(1+ i_P)^P$ by the end of the long period. Expressing this growth as a percentage increase provides the growth rate over the long period,

$$i_e = [X(1+ i_P)^P - X] / X \,. \qquad (7\text{-}4)$$

This is the effective interest rate over the long period, and it simplifies to *effective interest rate formula* that can be written either as

$$\boxed{i_e = (1+ i_P)^P - 1} \qquad (7\text{-}5)$$

or

$$i_e = (1+ r/P)^P - 1 \,. \qquad (7\text{-}6)$$

If the effective rate is known, then equation (7-5) indicates that

$$\boxed{i_P = (1+ i_e)^{1/P} - 1} \,, \qquad (7\text{-}7)$$

so the nominal rate r is $P\, i_P$. The following examples illustrate computing i_P and i_e and using them in factors. Nominal rates do not accurately measure the growth of capital or debt and *cannot* be used in the factors developed thus far in the text.

Example 7.1 *Effective Rates*

Table 7.1 shows the effective yearly interest rates corresponding to a nominal rate of 18% per year with different compounding periods. For example, 18% per year compounded monthly corresponds to rates of 1½% per month or 19.562% per year. ♣

Table 7.1 Compounding Periods

Period	i_P	i_e
Year	18.000% = 0.18 / 1	18.000% = $(1+ 0.18/1)^1 - 1$
Half-Year	9.000% = 0.18 / 2	18.810% = $(1+ 0.18/2)^2 - 1$
Quarter	4.500% = 0.18 / 4	19.252% = $(1+ 0.18/4)^4 - 1$
Month	1.500% = 0.18 / 12	19.562% = $(1+ 0.18/12)^{12} - 1$
Week	0.346% = 0.18 / 52	19.685% = $(1+ 0.18/52)^{52} - 1$
Day	0.049% = 0.18 / 365	19.716% = $(1+ 0.18/365)^{365} - 1$

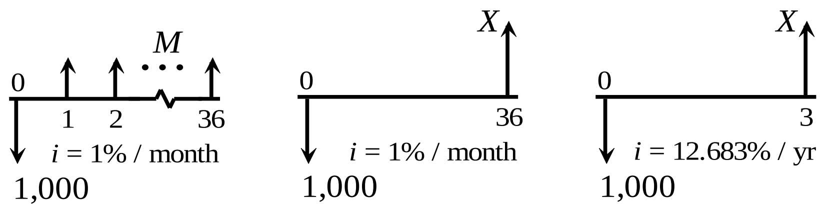

Figure 7.1 Rates and Cash Flow Diagrams

Example 7.2 *Using Rates in Factors*

Chablis borrows $1,000 at 12% per year compounded monthly. How much must she pay each month over a 3 year period? The nominal rate of 12% is *not* the rate at which her debt grows, so it cannot be used in the factors. Debt grows at a rate of $12\%/12$ or 1% per month, so the monthly payments M in Figure 7.1 are:

$$M = \$33.21 = 1{,}000\,(A|P,\ 1\%,\ 36\text{-}0) \qquad (7\text{-}8)$$

The cash flows occur monthly for this repayment method, so both the interest rate and time must be expressed in terms of months.

If she repays in one lump sum X after 3 years, then the problem can be solved in terms of months or years. In terms of months, her payment is given by:

$$X = \$1{,}430.77 = 1{,}000\,(F|P,\ 1\%,\ 36\text{-}0) \qquad (7\text{-}9)$$

If time is measured in terms of years, then the interest rate must be expressed on a yearly basis. The yearly interest rate is

$$12.683\% = (1 + 0.12/12)^{12} - 1\ , \qquad (7\text{-}10)$$

and the payment is

$$X = \$1{,}430.77 = 1{,}000\,(F|P,\ 12.683\%,\ 3\text{-}0) \qquad (7\text{-}11)$$

Notice that the cash flow diagram always must be expressed in terms of the interest rate used to solve the problem. ♣

7.2 Loans and Unknown Interest Rates

This section explains how to compute interest rates on loans and how to choose the best loan. Hundreds of the author's students have done projects that contrasted interest rates supplied by lenders with rates computed by the students, and the rates were rarely the same. There are several reasons for this. Lenders tend to quote nominal rather than effective rates, and they also use the APR formula that only approximates effective rates. More significantly, lenders and borrowers have different viewpoints.

Lenders generally seek to compute rates based on their profit, whereas borrowers want to include all cash flows. For example, a lender with a stated rate of 10% per year might issue a check to a borrower for $1,000 and require payment of $1,100 after 1 year. However, suppose that the lender charges $35 for administrative and risk related items, such as $10 for loan processing costs and $25 for a credit check. The borrower actually receives $965 $(1{,}000 - 35)$ at time 0 in exchange for $1,100 a year later, so the interest rate based on net cash flows is

$$13.99\% = (1{,}100 - 965)\,/\,965\ . \tag{7-12}$$

This lender implicitly defines interest as solely the payment ($100) for the use of capital ($1,000). If the administrative ($10) and risk related ($25) charges include a profit component for the lender, then this component typically is not reflected in the lender's rate. The borrower's definition recognizes all costs of borrowing, including administrative and risk related charges, in addition to the payment for the use of capital.

A prudent borrower examines the offerings of several lenders, seeks out *all* cash flows, and uses the net cash flow approach. Examples of costs of borrowing include credit checks, administrative costs, loss of cash discounts, additional legal fees, extra insurance, interest losses in non-interest-paying escrow accounts used to pay taxes or insurance, and so forth. The Truth in Lending law attempts to base the APR calculation on a net cash flow basis, but varying interpretations and practices make the APR unreliable, as well as its approximate nature. The following two examples illustrate the procedure for computing interest rates from the borrower's viewpoint and how to select the best loan.

Example 7.3 *Computing an Unknown Interest Rate*

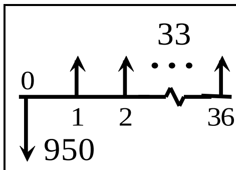

Figure 7.2 Loan

Cicero borrows $1,000 from Cataline. Credit checks and administrative costs result in Cicero's paying Cataline $50 at time 0. The note is $32 per month for 36 months, plus a monthly $1 loan insurance charge. What is the interest rate per year? The borrower's viewpoint recognizes all costs of the loan. Figure 7.2 shows that Cicero receives a net amount of $950 (1,000 - 50) at time 0, and pays $33 (32 + 1) per month. Using trial and error to solve the discounted amount formula

$$\$950 = 33\,(P\,|A,\, i_m,\, 36\text{-}0)\ , \tag{7-13}$$

yields a monthly rate i_m of 1.262%. The effective yearly rate is

$$16.243\% = (1 + 0.01262)^{12} - 1\ . \tag{7-14}$$

Notice that the borrower uses only net cash flows, and the rate from the lender's viewpoint is of no relevance to the borrower, other than satisfying curiosity. In this case, the lender might compute a monthly rate of 0.786%, the solution to

$$\$1{,}000 = 32\,(P\,|A,\, i_m,\, 36\text{-}0)\ , \tag{7-15}$$

and quote a rate of 9.429% (0.786% × 12) per year compounded monthly. ♣

Example 7.4 *Selecting the Best Loan*

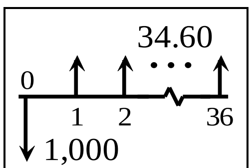

Figure 7.3
Alternative Loan

Consider the preceding example, but suppose that Cataline offers to finance the initial loan charges of $50 at 9.429% per year compounded monthly, increasing the monthly note from $33 to $34.60. What is the interest rate of this alternative from Cicero's viewpoint?

Cicero now receives a net amount of $1,000 at time 0 and pays $34.60 each month, as shown in Figure 7.3. How Cataline does his computations is not Cicero's concern unless his knowledge helps to negotiate more favorable terms. The monthly rate from Cicero's viewpoint is given by the solution to

$$\$1{,}000 = 34.60\,(P|A,\ i_m,\ 36\text{-}0)\ , \qquad (7\text{-}16)$$

so i_m equals 1.239%, and the effective yearly rate is

$$15.923\% = (1 + 0.01239)^{12} - 1\ . \qquad (7\text{-}17)$$

Financing the initial charges decreases the interest rate from 16.243% to 15.923%, but it requires paying interest on an additional $50 at a rate considerably more than Cicero can earn on his savings. He has an extra $50 (1,000 - 950) at time 0, but $1.60 less (34.60 - 33.00) each month to put into a savings account paying 6% per year compounded monthly. The compound amount of these changes is

$$-\$3.10 = 50\,(F|P, 0.5\%, 36\text{-}0) - 1.60\,(F|A, 0.5\%\ , 36\text{-}0)\ , \qquad (7\text{-}18)$$

so Cicero will have $3.10 *less* in savings at time 36 if he finances the $50. If Cicero's objective is to have the most money possible after 36 months, then he should not finance the extra $50, even though this lowers the interest rate on the loan. Choosing the lowest loan rate is a good approximate procedure or *heuristic* for selecting loans, but it might not maximize one's final balance. The next chapter pursues the topic of selecting the best alternative in more detail. ♣

7.3 Multi-Period Series

An understanding of nominal and effective interest rates contributes to determining unknown interest rates on loans, and it also is helpful for *multi-period series* in which the intervals between equal cash flows are longer than the compounding periods. Such series require the use of either an effective rate for the longer period or equivalent payments every compounding period, as illustrated by the following example.

Example 7.5 *Monthly Rate and Quarterly Cash Flows*

Chablis borrows $1,000 at 12% per year compounded monthly. If she makes quarterly payments over a 3 year period, then how much is each payment? Figure 7.4 shows two ways to solve this problem.

The method of measuring time in quarters and using the quarterly rate is shown on the left side of the figure. Use the effective interest rate formula with a quarter as the long period and a month as the short period to obtain the effective quarterly rate of 3.0301%,

$$3.0301\% = (1 + .01)^3 - 1, \qquad (7\text{-}19)$$

where 1% is the monthly rate and there are 3 months per quarter. Once the quarterly rate is known, then use capital recovery factor for the 12 quarterly payments:

$$\$100.64 = 1{,}000\,(A|P,\ 3.0301\%,\ 12\text{-}0)\ . \qquad (7\text{-}20)$$

The equivalents shown on the right side of the figure provide another solution procedure that will be useful later. A lender seeking to earn 1% per month would want monthly payments of $33.21:

$$M = \$33.21 = 1{,}000\,(A|P,\ 1\%,\ 36\text{-}0)\ . \qquad (7\text{-}21)$$

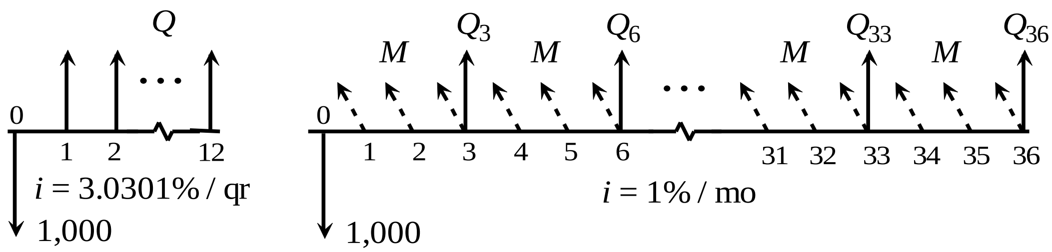

Figure 7.4 Monthly Rate, Quarterly Payments

The lender also would accept the compound amounts of these monthly payments at the end of each quarter since either the monthly or their compound amounts produce the same balance. The first compound amount Q_3 occurs at month 3, and its value is \$100.64:

$$Q_3 = \$100.64 = 33.21\,(F|A,\ 1\%,\ 3\text{-}0)\ . \tag{7-22}$$

The second compound amount Q_6 is at month 6, and its value is also \$100.64:

$$Q_6 = \$100.64 = 33.21\,(F|A,\ 1\%,\ 6\text{-}3)\ . \tag{7-23}$$

All compound amounts are the same because the last parameter of the sinking fund factor remains 3 whether it is calculated as 3-0, 6-3, …, 33-30, or 36-33, so

$$Q = \$100.64 = 33.21\,(F|A,\ 1\%,\ 3)\ , \tag{7-24}$$

as before. ♣

Example 7.6 *Monthly Rate and Yearly Cash Flows*

Multi-period series can be difficult to grasp initially, so consider the problem shown in Figure 7.5. If \$1,000 borrowed at 12% per year compounded monthly is repaid with three yearly notes, then how much is each payment? This is similar to the preceding example, except there are 12 compounding periods between payments instead of 3, so try to anticipate the logic.

The monthly rate is 1%, so the effective yearly rate is

$$12.6825\% = (1 + .01)^{12} - 1\ , \tag{7-25}$$

and the yearly payment is

$$\$421.24 = 1{,}000\,(A|P,\ 12.6825\%,\ 3\text{-}0)\ . \tag{7-26}$$

Alternatively, the monthly payment at 1% per month is

$$M = \$33.21 = 1{,}000\,(A|P,\ 1\%,\ 36\text{-}0)\ , \tag{7-27}$$

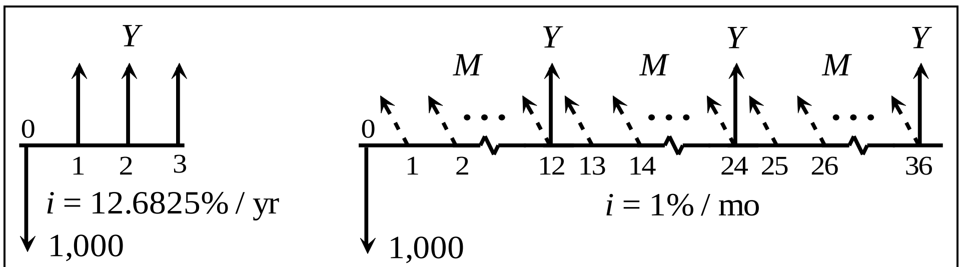

Figure 7.5 Monthly Rate, Yearly Payments

and a lender would accept the compound amount Y of 12 such monthly payments each year:

$$Y = \$421.24 = 33.21\,(F|A, 1\%, 12)\ . \quad (7\text{-}28)$$

The two procedures always give the same results if enough digits are used. ♣

7.4 Cash Flows Within Periods

Sometimes cash flows occur within the time period for which the interest rate is stated. For example, suppose that the effective interest rate of an account, i_e, is advertised as 9.381% per year, but it is known that monthly compounding is done. Solve the effective interest rate formula

$$i_e = (1 + i_P)^{P} - 1 \quad (7\text{-}29)$$

to express the rate for the short period i_P in terms of the long period using the *rate per period formula*:

$$\boxed{i_P = (1 + i_e)^{1/P} - 1}\ . \quad (7\text{-}30)$$

In this case, there are 12 compounding periods per year, so the monthly interest rate is

$$i_m = (1 + 0.09381)^{1/12} - 1 \quad (7\text{-}31)$$

or 0.75%. Now a cash flow diagram can be constructed that shows the cash flows each month, with an interest rate of 0.75% used in the factors.

If compounding does *not* occur within the period, then the timing of the cash flows must be adjusted to either the beginning or end of the period. For example, before computers modernized the banking industry, daily compounding generally was not feasible, and monthly compounding was more common. A deposit made during a month did not begin earning interest until the next month. Conversely, any withdrawals during a month reduced the amount that was compounded for that month. This policy of using adjusted dates for compounding purposes effectively forces all cash flows to occur on a periodic basis. It still is used by institutions that compound daily, but an investor loses only a day's interest instead of a month's interest.

Example 7.7 *Bonds*

An investor who wants to earn 10% per year is considering buying a bond with a coupon rate of 8% payable semiannually. The maturity period is 10 years, and the redemption value is \$3,000. What is the most that should be paid for the bond?

A coupon rate of 8% payable semiannually means that the rate is actually 4% per half-year, so a coupon of \$120 ($4\% \times 3{,}000$) is received every half-year. The cash flow diagram on the right must be in terms of half-years, since that is the frequency of the cash flows. The investor's effective discount rate is 10% per year, so the semi-annual rate is

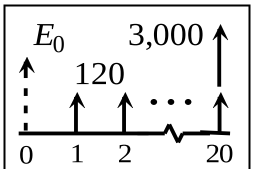

Figure 7.6 Bond

$$i_s = (1 + 0.10)^{1/2} - 1 \quad (7\text{-}32)$$

or 4.881%. The most that the investor should pay is the discounted amount,

$$E_0 = 120(P|A, 4.881\%, 20\text{-}0) + 3{,}000(P|F, 4.881\%, 20\text{-}0) \quad (7\text{-}33)$$

or \$2,667.32. ♣

Example 7.8 *Effect of Not Compounding Cash Flows Within a Period*

A company deposits \$10,000 of sales revenues at the end of each month in an account that has an effective interest rate of 9% per year, and it is known that monthly compounding occurs. The amount in the account after 5 years must be determined. One analyst mistakenly ignores the monthly compounding and models the cash flows as shown at the top of Figure 7.7, showing yearly deposits of \$120,000 with an interest rate of 9% per year. This results in a computed compound amount of \$718,165.27:

$$\$718{,}165.27 = 120{,}000\,(F|A,\ 9\%,\ 5\text{-}0) \qquad (7\text{-}34)$$

Another analyst correctly models the cash flows with monthly deposits of \$10,000 at 0.7207% per month,

$$0.7207\% = (1 + 0.09)^{1/12} - 1\ , \qquad (7\text{-}35)$$

to obtain a compound amount of:

$$\$747{,}328.69 = 10{,}000\,(F|A,\ 0.7207\%,\ 60\text{-}0) \quad ♣ \qquad (7\text{-}36)$$

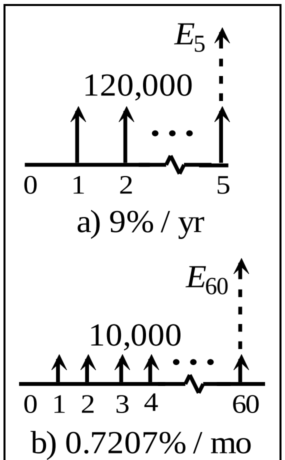

Figure 7.7 Flows within Periods

In the preceding example, the first analyst's simplification produced a result 3.9% smaller than the correct one. Repeating this process with an effective rate of 15% per year produces a result 6.3% too small. Nonetheless, project economics traditionally ignores compounding within a year, a practice that dates back to the use of slide rules. This is justified if approximate results are adequate, but modern computational aides such as calculators and spreadsheets can remove this source of error. This is particularly true in practice where economic calculations require relatively little time compared to obtaining the estimates of cash flows. In the classroom, most of the time is spent on computations, so the simplification of yearly cash flows seems reasonable.

7.5 Continuous Compounding

Some financial institutions offer continuous compounding, and continuous reinvestment is the general rule in industry. Table 7.1 shows effective rates as compounding progresses from a yearly to a daily basis, and the change in effective rates diminishes each time. For example, the effective rate increases from 18.000% to 18.810% when the compounding changes from yearly to semi-annually, but it only increases from 19.685% to 19.716% when compounding goes from weekly to daily. This section shows how to compute nominal, effective, and periodic rates as compounding becomes continuous, and then it illustrates the use of these rates in factors.

Nominal and Effective Rates

Each decrease in the length of the compounding period increases the number of compounding periods each year, until a limiting condition occurs. The formula for effective interest becomes

$$i_e = \lim_{P\to\infty} (1 + r/P)^P - 1\ , \qquad (7\text{-}37)$$

where P is the number of compounding periods. The periods for r and i_e are the same. For example, if r is a yearly nominal rate, then i_e is a yearly effective rate; if r is monthly, then i_e is monthly. The limit term equals e^r, so the effective rate for continuous compounding is:

$$\boxed{i_e = e^r - 1} . \tag{7-38}$$

For example, a yearly nominal rate of 18% compounded continuously implies a yearly effective rate for use in factors of 19.722%:

$$19.722\% = e^{0.18} - 1 . \tag{7-39}$$

This is slightly larger than the 19.716% rate for daily compounding shown in Table 7.1.

Sometimes the effective rate is known, and the nominal rate must be determined. Solving equation (7-38) for r results in

$$\boxed{r = ln(1 + i_e)} , \tag{7-40}$$

where ln is the natural logarithm function. As expected, a yearly effective rate of 19.722% corresponds to a yearly nominal rate of 18% compounded continuously:

$$18\% = ln(1 + 0.19722) . \tag{7-41}$$

Periodic Rates

Nominal rates r frequently are quoted on a yearly basis, such as 18% per year compounded continuously. If cash flows occur P times per year, such as monthly where P equals 12, then the rate must be expressed in terms of that period. Equation (7-38) indicates that the yearly effective rate is $e^r - 1$, and equation (7-30) provides the rate for the shorter period as

$$i_P = (1 + e^r - 1)^{1/P} - 1 \tag{7-42}$$

or

$$i_P = e^{r/P} - 1 . \tag{7-43}$$

The monthly rate that corresponds to a nominal yearly rate of 18% compounded continuously is:

$$i_m = 1.511\% = e^{0.18/12} - 1 . \tag{7-44}$$

This is a true monthly rate that can be used with factors for monthly cash flows. It results in the same effective yearly rate of 19.722% ($1.01511^{12} - 1$) that was computed earlier.

Equation (7-43) really is the same as equation (7-38), if the nominal rate is expressed as r / P per short period. For example, a continuous nominal rate of 18%/12 or 1½% per month corresponds to a monthly rate of

$$i_m = 1.511\% = e^{0.015} - 1 . \tag{7-45}$$

In general, continuous nominal rates can be re-expressed in terms of any period, such as 1½% per month, 4½% per quarter, 9% semiannually, 18% per year, 27% per year-and-a-half, or 36% biennially, without affecting the same effective rate or any computations. This is not the case for discrete compounding where different formulas are used.

Example 7.9 *Rates for Different Periods*

Table 7.2 converts a nominal rate of 12% per year compounded continuously into rates that can be used in factors for cash flows at different discrete intervals. It does this by expressing the 12% per year as 1% per month, 3% per quarter, and so forth. Note the rates in the table also are valid for an effective rate of 12.7497% where compounding is continuous, since equation (7-40) indicates that *ln*(1.127497) equals 12%. ♣

Table 7.2 Rates per Period

Period	Rate per Period
Monthly	$i_m = 1.0050\% = e^{0.01} - 1$
Quarterly	$i_q = 3.0455\% = e^{0.03} - 1$
Semiannually	$i_s = 6.1837\% = e^{0.06} - 1$
Yearly	$i_y = 12.7497\% = e^{0.12} - 1$
Biennially	$i_b = 27.1249\% = e^{0.24} - 1$

Factors

Periodic rates such as those computed above accurately measure the growth of capital or debt, so they can be used in all of the single payment, uniform series, and gradient factors developed thus far. For example, if interest is 12% per year compounded continuously, then the discounted amount of \$1,000 occurring 8 months hence is given by:

$$\$923.12 = 1{,}000(P|F,\ 1.0050\%,\ 8\text{-}0) \qquad (7\text{-}46)$$

An easier approach is to develop special factors for continuous compounding. If r is the nominal continuous rate per period (e.g., per month, per quarter, etc.), then the periodic rate is $e^r - 1$. Substituting $e^r - 1$ for i in each factor's formula yields the continuous compounding factors. Single payment factors are used more commonly than the others:

$$\boxed{(P|F, r, m) = e^{-r\,m}} \qquad (7\text{-}47)$$

and

$$\boxed{(F|P, r, m) = e^{r\,m}} \qquad (7\text{-}48)$$

Continuous compounding allows cash flows to occur at any time, so m does not have to be an integer. However, it is necessary for m and r to be dimensionally consistent, so one cannot be in terms of months with the other measured in years.

Example 7.10 *Continuous Compounding Single Payment Factors*

Consider the cash flows in Figure 7.8 at months 1.5, 3, 7, and 9.2. The interest rate is 12% per year compounded continuously. What is equivalent at time 5? The factors given above easily can be evaluated using a calculator using a monthly nominal rate of 1%, so

$$E_5 = 11\,e^{\,0.01\,(5\text{-}1.5)} + 13\,e^{\,0.01\,(5\text{-}3)} + 17\,e^{-0.01\,(7\text{-}5)} + 18\,e^{-0.01\,(9.2\text{-}5)}\ , \qquad (7\text{-}49)$$

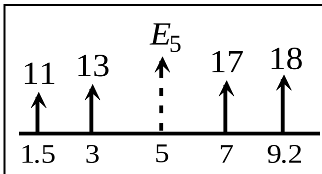

Figure 7.8 Continuous Compounding Factors

or \$58.58 will produce the same balance as the original cash flows. ♣

7.6 Continuous Cash Flows

Several situations can be modeled using continuous cash flows, such as transactions involving automatic teller machines or on-line credit card purchases of gasoline. This model also can be applied to industrial operations and maintenance expenses that occur fairly uniformly during the year. This section shows how to reduce a continuous cash flow to a single payment that can be manipulated as necessary using the continuous single payment factors.

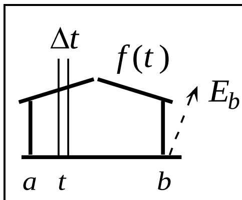

Figure 7.9 Funds Flow Equivalent

Suppose that funds flow at a rate of $f(t)$, such as \$10,000 per year. Then the amount of funds that flow during some small interval at time t is approximately $f(t)\Delta t$, as shown in the figure on the right. For example, if Δt should be 1 day or 1/365 of a year, then the cash flow for Δt would be \$10,000(1/365). If $f(t)$ is a constant, then $f(t)\Delta t$ is an exact value, otherwise it is approximate. The compound amount at time b is $f(t)\Delta t\, e^{r(b-t)}$. The sum of the compound amounts for all intervals Δt between a and b is the equivalent of the entire funds flow at time b. Calculus provides this sum as

$$E_b = \int_a^b f(t) e^{r(b-t)} dt \ , \tag{7-50}$$

where a, b, $f(t)$, and r are all dimensionally consistent. For example, they must all be expressed in terms of years or all in terms of months. The most commonly occurring case is when $f(t)$ is some constant K, such as \$10,000 per year, where the expression for the equivalent simplifies to:

$$\boxed{E_b = K \frac{e^{r(b-a)} - 1}{r}} \tag{7-51}$$

Example 7.11 *Constant Funds Flow Function*

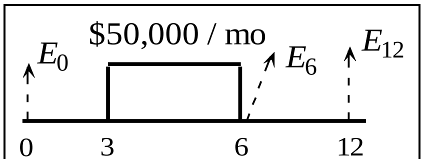

Figure 7.10 Constant Funds Flow Function

All transactions at a new service station require inserting a credit card into a pump, so that payment automatically is credited to an account paying 6% per year compounded continuously. Revenues are expected to be \$50,000 per month from month 3 through month 6. How much would have to be received at month 0 to produce the same balance? How much will be in the account at month 12?

The first step in solving this problem is to reduce the funds flow to a single equivalent at month 6 using equation (7-51). Dimensional consistency requires expressing the nominal interest rate as ½% per month, and then

$$E_6 = 50{,}000 \frac{e^{0.005(6-3)} - 1}{0.005} \tag{7-52}$$

or \$151,130.65. Thereafter, single payment factors can be used to compute

$$E_0 = 151{,}130.65e^{-0.005(6-0)} \tag{7-53}$$

or $146,664.06 and

$$E_{12} = 151{,}130.65e^{0.005(12-6)} \tag{7-54}$$

or $155,733.26. ♣

Example 7.12 *Discrete Approximations*

Cash flows that are actually continuous frequently are treated as if they occurred at the end of a period. For example, consider a continuous cash flow of $365,000 per year. Suppose that the effective interest rate is 10% per year and that compounding is continuous during the year. Then its actual compound amount after a year can be computed first by determining the nominal rate per year,

$$r = ln(1.10) \tag{7-55}$$

or 9.531%, and then using equation (7-51)

$$CA = 365{,}000\frac{e^{0.09531(1-0)}-1}{0.09531} \tag{7-56}$$

to obtain $382,960.14. Table 7.3 shows the compound amounts caused by assuming end-of-period flows. For example, the compound amount associated with daily flows assumes $1,000 per day at a daily interest rate given by equation (7-43) as

$$i_d = e^{0.09531/365} - 1 \tag{7-57}$$

or 0.02612%. These $1,000 flows compound to 1,000($F|A$, 0.02612%, 365-0) or $382,910.15 after a year, a 0.013% error. Similarly, a 52-week and 12-month year are used to compute the compound amounts for those periods. The end-of-year assumption typically used in the classroom introduces a 4.69% error. ♣

Table 7.3 Discrete Approximations

Period	*CA*	% Error
Actual	$382,960.14	0.000%
Day	382,910.15	0.013
Week	382,592.65	0.096
Month	381,374.37	0.414
Quarter	378,254.04	1.229
Half-Year	373,697.05	2.419
Year	365,000.00	4.690

7.7 Summary

This chapter first explains how to convert a nominal rate such as 18% per year compounded monthly to the actual monthly rate of 1½% per month. This monthly rate can be used in factors, or it can be used to compute an effective rate or yield for a longer period, such as a year, using the effective interest rate formula in equation (7-5).

Lenders tend to quote nominal rates rather than effective rates. Moreover, these nominal rates frequently are based only on cash flows that the lender claims a profit. For example, the quoted rate might not include costs that a borrower must pay for a credit check, administrative charges, or insurance required as a condition of the loan. A prudent borrower always draws the cash flow diagram from his or her own perspective, and then computes the rate per compounding period. If the period is less than a year, such as a month, then it might be desirable to determine the effective rate per year.

The next topic is multi-period series, such as a yearly series when compounding occurs on a monthly basis. There are two approaches to solving problems of this type. The first is to compute the effective rate for the time interval between cash flows. This typically results in a rate that is not in the tables, so the series factors must be computed with a calculator. The second method is to replace each original series flow with its equivalent series having an interval equal to the compounding period.

Sometimes cash flows occur more frequently than the quoted interest rate. If the quoted rate is nominal, then the rate for a shorter compounding period equals the quoted rate divided by the number of shorter periods within the nominal period. If the quoted rate is effective, then use the rate per period formula in equation (7-30). If cash flows occur more frequently than the stated compounding frequency, then they are not credited to the account until the beginning of the next period.

Continuous compounding allows cash flows to earn interest at any time, not just at the beginning or end of compounding periods. If cash flows should occur at regular intervals, then equation (7-43) converts continuous rate into an effective rate per period that can be used in previously developed factors. Single payments occur frequently enough to warrant developing formulas for the continuous compounding single payment factors.

The compound amount of continuous funds flows can be determined by integrating the funds flow function multiplied by the single payment compound amount factor over the region of the flow. The most common case where this occurs is when the funds flow function is a constant, and equation (7-51) provides a convenient formula for the compound amount at the end of the flow. The continuous compounding single payment factors allow equivalents to be computed at other times.

Questions

Section 1: Nominal and Effective Interest

1.1 An account pays 12% per year compounded monthly. What interest rate should be used in factors for cash flows that occur on a monthly, quarterly, semi-annual, annual, and biennial basis? (1%, 3.03010%, 6.15202%, 12.68250%, 26.97346%)

1.2 A bank advertises that its yield on certificates of deposits is 7.8% per year. If compounding is monthly, then what is the yearly nominal rate? (7.5343%)

Section 2: Loans and Unknown Interest Rates

2.1 Horace is considering a chariot loan that Catullus quotes at a rate of 9% per year compounded monthly. The credit check and loan origination fee cost $200. If Horace borrows $14,000 with a 36 month repayment period, then his monthly principal and interest payment will be $445.20. Catullus also requires a credit life insurance policy costing $2.25 monthly that will pay off the loan if Horace reaches his expiration date prior to paying off the loan, plus extra collision insurance beyond what he would have purchased. This extra insurance costs $150 per year, payable at the beginning of each year. There are also costs such as recording the title and so forth that amount to $50; these costs would have to be paid even if the car were not financed. From Horace's viewpoint, what is the effective yearly interest rate? (13.0966%)

2.2 Suppose that the lender in question 2.1 offers to finance the $350 for credit check, origination fee, and initial extra insurance payment in exchange for increasing the monthly principal and interest payment by $11.13. What would the interest rate be on this loan? Should Horace accept this offer of additional financing if he can put his money into an account paying 6% per year compounded monthly? (13.0036%, reduce balance by $18.97 if finance $350)

Section 3: Multi-Period Series

3.1 A loan of $1,000 is to be repaid with quarterly payments over 5 years. The interest rate is 8% per year compounded monthly. What are the payments? ($61.24)

3.2 Suppose that the interest rate is 9% per year, compounded monthly. Deposits of $1,000 are made at years 3, 4, and 5, and then equal withdrawals are made at years 10, 11, and 12 resulting in a final balance of $0.

a) What is the effective interest per year? (9.38069%)
b) Use the effective interest rate to compute the withdrawals. ($1,873.20)
c) What is the rate per month? (0.75%)
d) Use monthly equivalents to compute the withdrawals. ($1,873.20)
e) Use monthly equivalents to compute the withdrawals if the interest rate changes immediately after the last deposit to 12% per year compounded monthly. ($2,234.50)

3.3 Withdrawals of $1,000 are planned at months 30, 42, 54, and 66. The interest rate is 12% per year compounded monthly. Use monthly equivalents of the withdrawals to determine how much must be deposited today. ($2,503.20)

Section 4: Cash Flows Within Periods

4.1 An investor wants to earn a yield of 9% from a $10,000 bond with a coupon rate of 6% payable semiannually. The bond's life is 10 years, and it was issued 4 years ago. The eighth payment will be made immediately after the purchase. What is the maximum that the investor should pay for the bond? ($9,013.48)

4.2 A company deposits $10,000 of sales revenues at the end of each month in an account that has an effective interest rate of 12% per year, and it is known that monthly compounding occurs. Determine the amount in the account after 5 years and the percent error introduced by assuming a single cash flow of $120,000 at the end of the year, where Percent Error = (Approximate Value – True Value) / True Value. ($803,412.71, -5.112% or 5.112% low)

4.3 A savings program pays 6% per year compounded monthly. However, it credits deposits during the month as if they occurred at the end of the month, and it treats withdrawals during the month as if they occurred at the beginning of the month. Joe McSave makes 6 mid-month deposits of $100, followed by 6 mid-month withdrawals of $40. How much is in the account at the end of 12 months? ($381.77)

Section 5: Continuous Compounding

5.1 An account pays 10% per year compounded continuously.

a) What is its nominal monthly rate? (0.83333%)
b) What is its effective monthly rate? (0.83682%)
c) What is its effective yearly rate based on the nominal yearly rate? (10.51709%)
d) Use the effective yearly rate to compute the effective monthly rate. It should equal the previously computed value. (0.83682%)

e) If deposits of \$100 are made at the end of months 1, 2, …, 12, then how much is in the account at the end of month 12? (\$1,256.80)

5.2 If a certificate of deposit's yearly effective rate is 8% and compounding is continuous, then what is its nominal yearly rate? (7.69610%)

5.3 Interest is 9% per year compounded continuously, and cash flows of \$100 are made at years 6, 7.25, 12.5, and 14. What single cash flow at time 10 is equivalent? (\$421.03)

Section 6: Continuous Cash Flows

6.1 Funds flow at a constant rate of \$120,000 per year, and interest is 9% per year compounded continuously. Consider the funds that flow from month 3.5 to month 9.0.

a) What is their discounted amount at month 1? (\$52,880.21)

b) What is their compound amount at month 19.5? (\$60,750.73)

6.2 Other types of flow rates also can occur in addition to the constant funds flow. For example, sometimes revenues decline exponentially over time. This leads to a funds flow function $f(t)$ of the form Ve^{-wt}. Using this function in equation (7-50) results in the following compound amount:

$$E_b = V\frac{e^{r(b-a)-wa} - e^{-wb}}{w+r} \tag{7-58}$$

As an example, suppose that the flow rate of an oil well at year t is $5{,}000e^{-0.10t}$ barrels per year. If oil sells for \$20 per barrel, then the funds flow rate function is $\$100{,}000e^{-0.10t}$ per year. What is the compound amount at month 18 of a flow from months 6 to 18 (years 0.5 to 1.5) if the interest rate is 8% per year compounded continuously? (\$94,303.64)

6.3 Compute the percent errors for Table 7.3 if the effective interest rate is 15%, but compounding is continuous? (0.0000%, 0.0191%, 0.1343%, 0.5812%, 1.7369%, 3.4534%, 6.8254%)