5: Uniform Series

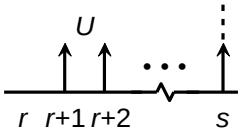
- Cash flows of uniform series
  - Equal
  - Occur each compounding period
- Also known as *annuities*, even if not yearly
- Use one series factor instead of several single payment factors

5.1 Compound Amounts

- Two situations
  - Given the cash flows, determine the compound amount
  - Given the compound amount, determine the cash flows

Uniform Series Compound Amount Factor

$E_S = U\,(F\,|\,A, i, s\text{-}r)$



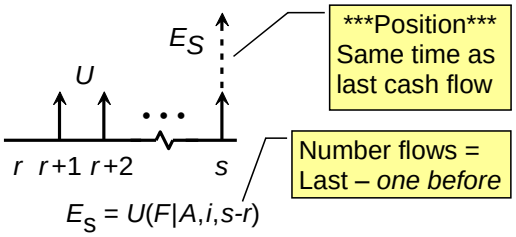
$$E_S = U\gamma^{s-(r+1)} + U\gamma^{s-(r+2)} + \ldots + U$$

$$E_S = U\,[\,\gamma^{s-(r+1)} + \gamma^{s-(r+2)} + \ldots + 1]$$

$$E_S = U\,[\,(\gamma^{s-r} - 1) / i\,]$$

$$\boxed{E_S = U\,(F|A, i, m = s\text{-}r)}$$

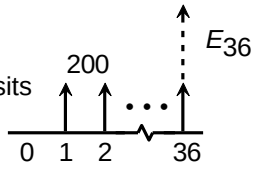
Observations



- Uniform series CA factor or "*F* given *A*"
  - Find a ***f***uture CA given an ***a***nnuity, interest rate, and number of series flows

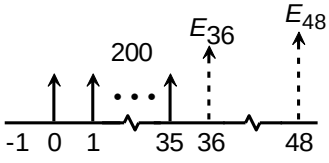
Example 5.1 Uniform Series CA

- Purchase a car in 3 years
  - Save $200 per month
  - 36 end-of-month deposits
  - i = 1/2 % per month
- Amount in savings
  - E_{36} = $7,867.22 = 200 ($F|A$, 1/2%, 36-0)

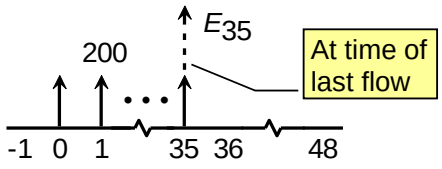


Example 5.2 Delayed CA

- 36 deposits of $200 at 1/2% / mo
  - Beginning-of-month deposits
- Savings at t = 36 and t = 48?
- CA *must* be at time of last flow to use F|A, so multi-step problem



First Determine E_{35}



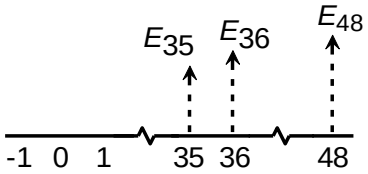
Last

E_{35} = \$7,867.22 = 200 ($F|A$, 1/2%, 35 − (-1))

- Have \$7,867.22 in savings at t = 35
  - No longer need original cash flows

One before

Then Other Equivalents



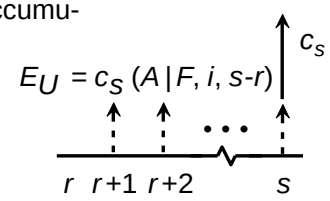
E_{36} = \$7,906.56 = $E_{35}(F|P$, 1/2%, 36 − 35)

To − From

E_{48} = \$8,394.32 = $E_{35}(F|P$, 1/2%, 48 − 35)

Sinking Funds

- Equal deposits to accumulate a given CA
  - Given CA, find prior flows
- Relationship of CA to its prior deposits known from $F|A$



$$\text{CA} = \text{US} \times [\,(\gamma^{s-r-1})\,/\,i\,]$$

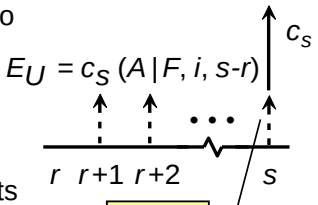
$$\text{US} = \text{CA} \times [\,i\,/\,(\gamma^{s-r-1})\,]$$

$$E_U = c_S[\,i\,/\,(\gamma^{s-r-1})]$$

$$\boxed{E_U = c_S(A|F, i, s\text{-}r)}$$

Observations

- Sinking fund factor also Known as "A|F"
  - Find prior ***a***nnuity given ***f***uture CA, ***i***nterest rate, and number of payments
- $E_U = c_S(A|F, i, s\text{-}r)$

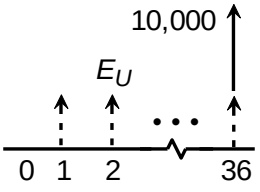


At time of CA

Last minus ***One Before***

Example 5.3 End-of-Month Sinking Fund

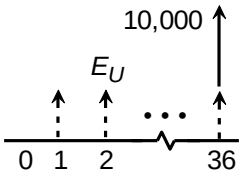
- Want \$10,000 for car in 3 years
  - 36 end-of-month deposits
  - i = 1/2 % / mo
- Amount of each deposit?



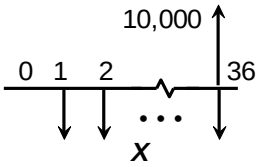
E_U = \$254.22 = 10,000 ($A|F$, 1/2%, 36-0)

Different Diagrams − Same Meaning

- Preceeding diagram shows *equivalents*
  - Dashed, same direction

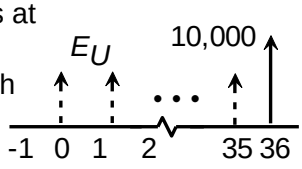


- Could show deposit *flows*
  - Solid lines
  - Depositor's viewpoint
    - Deposits flow *from*
    - Withdrawal flows *to*



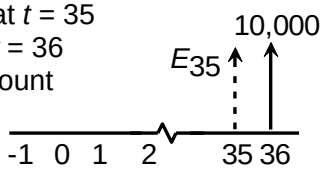
Example 5.4 Beginning-of-Month Sinking Fund

- Still make 36 deposits at 1/2 % per month
  - Beginning-of-month
- Need CA at $t = 35$ to use $A|F$
- So ask, "How much is needed at time 35 to have \$10,000 at time 36?"
  - If diagram does not *exactly* match diagram used to derive factor, then ***multi-step problem!***



First Step

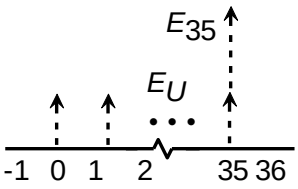
- Amount necessary at $t = 35$ to have \$10,000 at $t = 36$ is its discounted amount



E_{35} = \$9,950.25 = 10,000 ($P|F$, 1/2%, 36-35)

Second Step

- What deposits at 0, 1, 2, …, 35 compound to E_{35} = \$9,950.25?



E_U = \$252.95 = E_{35} ($A|F$, 1/2%, 35 – (-1))

Careful!

Challenges

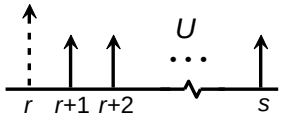
- Position of equivalents
- Fourth parameter

5.2 Discounted Amounts

- Two situations
  - Given the cash flows, determine the discounted amount
  - Given the discounted amount, determine the cash flows

Uniform Series DA Factor

$E_r = U\,(P \mid A, i, s\text{-}r)$



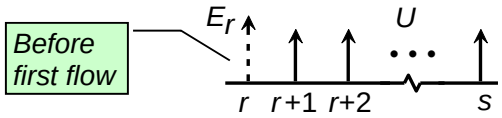
$$E_r = U\gamma^{-(r+1-r)} + U\gamma^{-(r+2-r)} + \ldots + U\gamma^{-(s-r)}$$

$$E_r = U\,[\,\gamma^{-1} + \gamma^{-2} + \ldots + \gamma^{-(s-r)}]$$

$$E_r = U\,[\,1 - (1+i)^{-(s-r)}\,]\,/\,i$$

$$\boxed{E_r = U\,(P|A, i, s\text{-}r)}$$

Observations



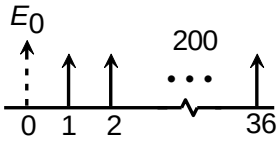
$Er = U\,(P\,|\,A,\, i,\, s\text{-}r)$

Number of flows = Last – *One Before*

- Uniform series discounted amount factor
- "P | A" since looking for ***p***rior given ***a***nnuity

Example 5.5 Uniform Series DA

- Borrow for car
  - Afford \$200 / mo
  - 36 end-of-month payments
  - i = 3/4 % / mo

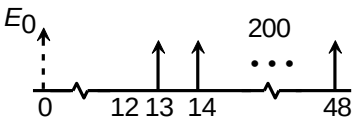


- How much car?
  - Amount loaned = Discounted payments

  E_0 = \$6,289.36 = 200($P\,|\,A$, 3/4%, 36-0)
- Have \$1,577.86 (7,867.22 - 6,289.36) more for car by saving \$200 / mo instead

Example 5.6 DA with Delayed Series

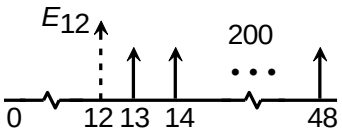
- Still 36 notes of \$200 at 3/4% / mo
  - Delay for 12 months



- Determine amount borrowed
- $P\,|\,A$ requires equivalent to be before first cash flow, so multi-step problem

First Step

- What must the debt be at t = 12 to require payments of \$200?

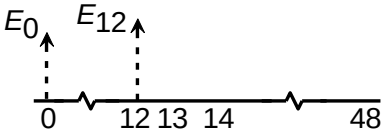


  - Time 12 chosen so that $P\,|\,A$ can be used
  - One period before first flow!

E_{12} = \$6,289.36 = 200($P\,|\,A$, 3/4%, 48-12)

Second Step

- If the debt is E_{12} at time 12, then what was it at time 0?



E_0 = \$5,749.97 = E_{12} ($P\,|\,F$, 3/4%, 12-0)

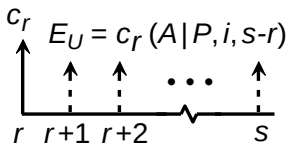
- Now can borrow only \$5,749.97
- The incredible shrinking car

The Moral to These Examples Is ...

- Borrowing instead of saving and delaying payments decreases the amount available for a purchaser

Capital Recovery Factor

- At what rate must invested capital be recovered to earn a specified rate of return?



- Know from deriving P/A

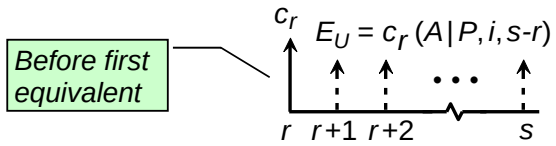
$$DA = US \times [\,1 - (1+i)^{-(s-r)}] \,/\, i$$

$$US = DA \times i \;/\; [\,1 - (1+i)^{-(s-r)}]$$

$$E_U = c_r \times i \;/\; [\,1 - (1+i)^{-(s-r)}]$$

$$\boxed{E_U = c_r(A|P\ i,\ s\text{-}r)}$$

Observations



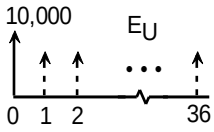
$$E_U = c_r(A|P\ i,\ s\text{-}r)$$

Number of flows = Last – One Before

- (Uniform series) Capital recovery factor
- "*A* given *P*" since finding an equivalent ***a***nnuity given a ***p***rior cash flow

Example 5.7 Capital Recovery

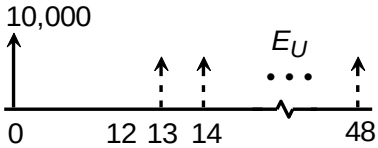
- Borrow \$10,000 to buy a car
  - 36 EOM notes
  - i = 3/4 % / mo
- Payments that provide a return of 3/4% / mo?



$$E_U = \$318.00 = 10{,}000(A|P, 3/4\%, 36\text{-}0)$$

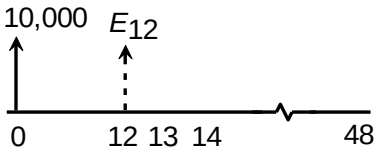
- Saved \$10,000 with 36 deposits of \$254.22, so \$63.78 (318.00 – 254.22) more each mo

Example 5.8 CR with Delayed Payments



- Same loan but delay payments by 12 months
- Determine payments
  - $A|P$ requires equivalent to be one period before first cash flow, so multi-step problem

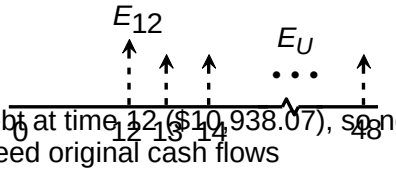
First Step



- If \$10,000 owed at time 0, then how much owed at time 12?
- Time 12 chosen one period before notes so $A|P$ can be used in step 2 ☺

$$E_{12} = \$10{,}938.07 = 10{,}000(F|P, 3/4\%, 12\text{-}0)$$

Second Step



- Know debt at time 12 (\$10,938.07), so no longer need original cash flows
  - Note position of E_{12}, one before payments

$$E_U = \$347.83 = E_{12}(A|P, 3/4\%, 48\text{-}12)$$

- Delay costs \$94.88 (347.83 – 252.95) more per month than saving for the car

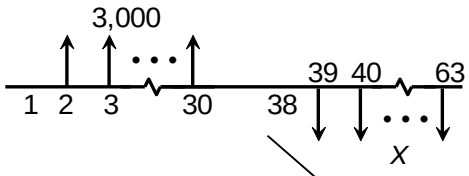
All Together Now ...

- Borrowing is BAD

5.3 Multiple Series

- Series of deposits followed by series of withdrawals fairly common
- Strategy
  - Use $F|A$ or $P|A$ on known series
  - Use $F|P$ or $P|F$ on single equivalent
    - Move to either last flow or one period before first flow of unknown series
  - Use $A|F$ or $A|P$ for unknown series
- Number flows = First - One before

Ex 5.9 Known Deposits, Unknown Withdrawals

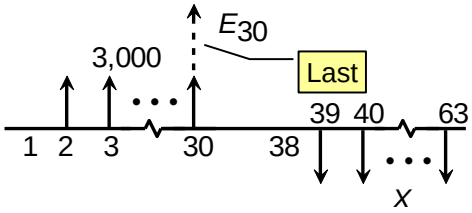


Drawn from account's perspective

- Start deposits in 2 years
  - $3,000 @ 8% until $t = 30$
  - College from $t =$ 31 to 38.
- Retire on X from $t = 39$ to 63?
- First question?

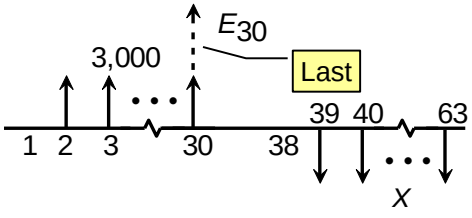
First Step a)

- How much in savings at $t = 30$?



First Step b)

- How much in savings at $t = 30$?



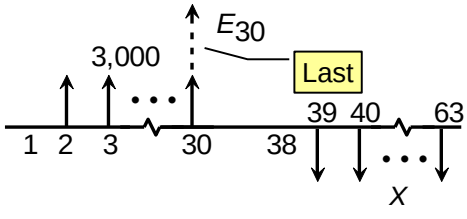
$E_{30} =$ \$311,897.81 = 3,000($F|A$, 8%, 30-1)

Last

One *before*

First Step c)

- How much in savings at $t = 30$?



$E_{30} =$ \$311,897.81 = 3,000($F|A$, 8%, 30-1)

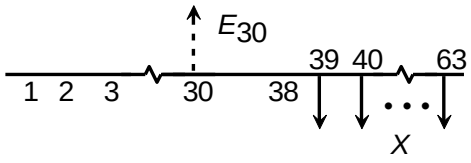
Last

One *before*

- No longer need original flows

First Step d)

- Original problem is now

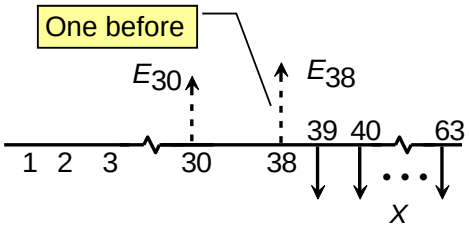


E_{30} = \$311,897.81 = 3,000($F|A$, 8%, 30-1)

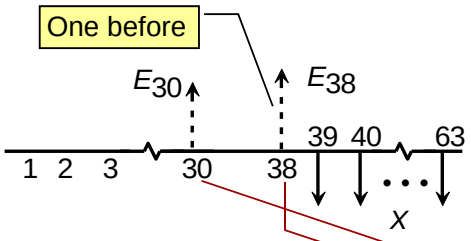
- Next question?

Second Step a)

- How much in savings at t = 38?



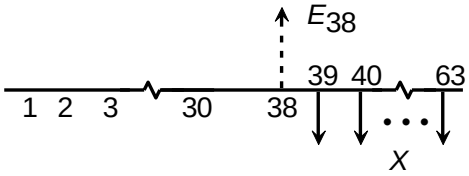
Second Step b)



E_{38} = \$577, 301.08 = $E30$ ($F|P$, 8%, 38-30)

Future – Prior

Second Step c)

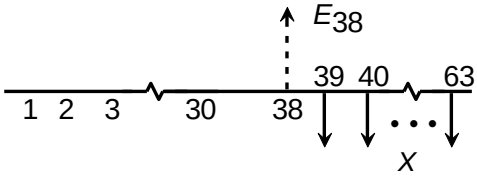


E_{38} = \$577, 301.08 = $E30$ ($F|P$, 8%, 38-30)

- Next question?

Third Step

- Withdrawals?



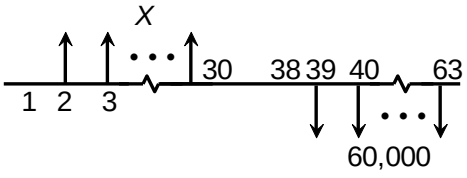
X = \$54,080.86 = $E38$ ($A|P$, 8%, 63-38)

Last

One before

Ex 5.10 Known Withdrawals, Unknown Deposits

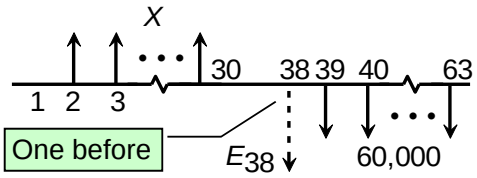
- As before, but want withdrawals of \$60,000



- Deposits?
- What is the first question?

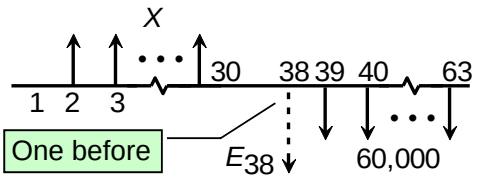
First Step a)

❖ What must be in savings at $t = 38$?



First Step b)

❖ What must be in savings at $t = 38$?



E_{38} = \$640,486.57 = 60,000 ($P|A$, 8%, 63-38)

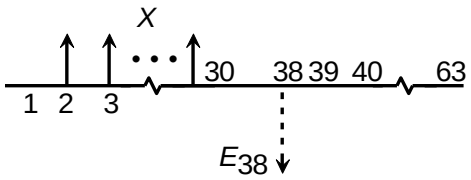
❖ Do not need original flows

Last

One before

First Step c)

❖ What must be in savings at $t = 38$?



E_{38} = \$640,486.57 = 60,000 ($P|A$, 8%, 63-38)

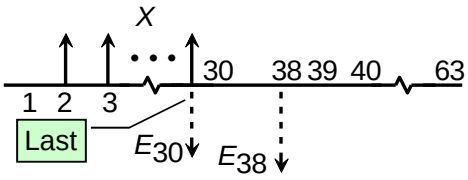
❖ Next question?

Last

One before

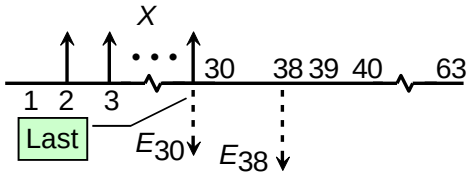
Second Step a)

❖ What must be in savings at $t = 30$?



Second Step b)

❖ What must be in savings at $t = 30$?

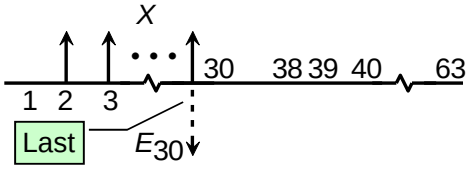


E_{30} = \$346,034.97 = $E38$ ($P|F$, 8%, 38-30)

Future – Prior

Second Step c)

❖ What must be in savings at $t = 30$?

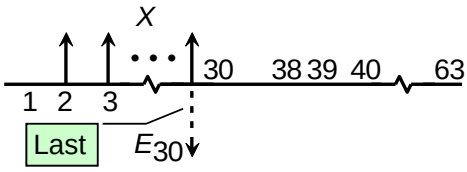


E_{30} = \$346,034.97 = $E38$ ($P|F$, 8%, 38-30)

❖ Next question?

Third Step

- Deposits?



X = \$3,328.35 = E_{30} (A|F, 8%, 30-1)

Last

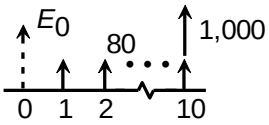
One before

5.4 Bonds

- Government and private industry borrow directly from the public
  - Bypass banks
- Contract to make future payments
  - Sold on open market
  - Max price = Discounted payments
- Terminology

Terminology

- *Maturity date*: time of final payment (10)
- *Coupon*: series paid to bondholder (\$80)
- *Redemption, face, or par value*: paid at maturity date in addition to coupon (\$1,000)
- *Coupon rate*: rate used to compute the series payments (8%)

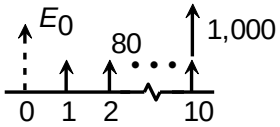


Coupon = Face Value × Coupon Rate

80 = 1,000 × 8%

- Max to pay (E_0)

Example 5.11 Purchase of a New Bond



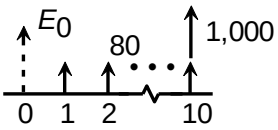
- Max to pay to earn 9%?

E_0 = \$935.82 = 80(P|A, 9%, 10-0)

+ 1,000 (P|F, 9%, 10-0)

  - Offer (\$935.82) is *below par* (\$1,000) when IRR > Coupon Rate

Purchase @ 8%



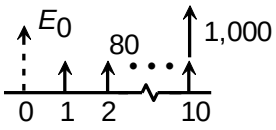
- Max to pay to earn 8%?

E_0 = \$1,000 = 80(P|A, 8%, 10-0)

+ 1,000 (P|F, 8%, 10-0)

  - Offer (\$1,000) is *at par* (\$1,000) when IRR = Coupon Rate

Purchase @ 7%



- Max to pay to earn 7%?

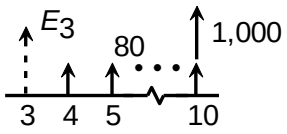
E_0 = \$1,070.24 = 80(P|A, 7%, 10-0)

+ 1,000 (P|F, 7%, 10-0)

  - Offer (\$1,070.24) is *above par* (\$1,000) when IRR < Coupon Rate

Example 5.12 Purchase of an Existing Bond

- Max to pay at time 3 for same bond if third coupon just paid to earn an IRR of 9%



E_3 = \$949.67 = 80 ($P|A$, 9%, 10-3)
+ 1,000 ($P|F$, 9%, 10-3)

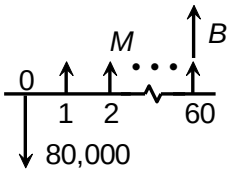
- Offer (\$949.67) is *below par* (\$1,000) when IRR > Coupon Rate

5.5 Loans

- Balloon notes
  - Final payment larger or "balloons"
- Principal and interest
  - Components of each payment
- Early repayment
  - Effect on duration of loan

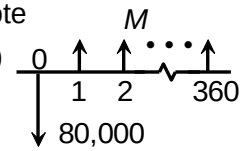
Example 5.13 Balloon Note

- Borrow \$80,000 for house
  - 3/4% / mo for 5 years
  - M as for 30 yr loan
  - Balance B due @ t = 60



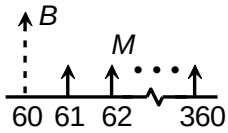
- First, determine monthly note

M = 80,000 ($A|P$, ¾%, 360-0)
= \$643.70



Step 2

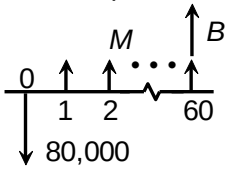
- Second, determine balloon payment
  - At time 60, with payment 60 just made
  - Payments 61, …, 360 remain
  - Easiest to discount remaining payments



B = \$76,704.11 = 643.70 ($P|A$, ¾%, 360-60)

Alternative Step 2

- Could use balance equation once M known



0 = 80,000 ($F|P$, ¾%, 60-0)
– 643.70 ($F|A$, ¾%, 60-0) – B

$\Rightarrow$ B = \$76,704.11, as before

- Discounting easier

Principal and Interest

- Two parts of each note or payment

Loan Payment = Interest + Principal

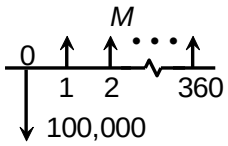
  - Principal component reduces the debt
  - Interest reduces individual's income tax
    - Home loan
    - Business loan
- Interest on several payments

Σ*Interest* = Σ*Loan Payments* – *Debt Reduction*

  - Easier than period-by-period balances

Example 5.14 Principal and Interest Payment

- House loan for \$100,000
  - 3/4% / mo
  - 30 years
- Notes 3, 4, …, 14 in next tax year. Interest?
- First, determine the monthly payment
  M = \$804.62 = 100,000 ($A|P$, ¾%, 360-0)



$\boxed{\Sigma Interest = \Sigma Loan\ Payments - Debt\ Reduction}$

$\Sigma Loan\ Payments$ = \$9,655.44 = 804.62 × 12

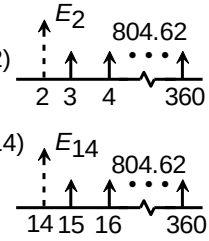
Debt Reduction = Debt before – Debt after

Step 2

Before = E_2 = \$99,890.35
= 804.62 ($P|A$, ¾%, 360-2)

After = E_{14} = \$99,196.86
= 804.62 ($P|A$, ¾%, 360-14)

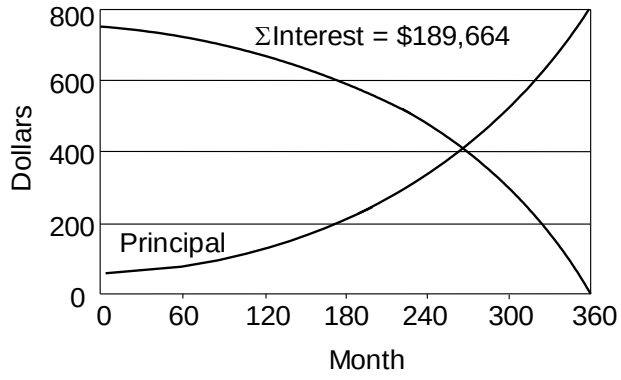
Reduction = Before – After
= \$693.48



$\Sigma Interest = \Sigma Loan\ Payments - Debt\ Reduction$
= 9,655.44 – 693.48

$\Sigma Interest$ = \$8,961.96

Principle & Interest Components over Time

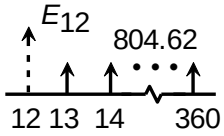


Early Repayment

- Notes on a loan
  - Always pay interest first
  - Remainder reduces principal
- Extra payment reduces the debt
- Amount of legally required note not changed
- Number of payments is reduced
- Some lenders allow, others charge penalty

Example 5.15 Early Loan Repayment

- Same loan
  - \$100,000 @ 3/4%
  - 360 notes of \$804.62
- Pay \$10,000 extra at t = 12



Before = 804.62 ($P|A$, ¾%, 360-12)
= 99,316.48

After = \$89,316.48 = 99,316.48 – 10,000

- \$804.62 / mo continues until time s when
  \$89,316.48 = DA of Remaining Payments

Step 2

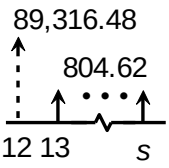
$89{,}316.48 = 804.62(P|A, ¾\%, s\text{-}12)$

$(P|A, ¾\%, s\text{-}12) = 111.005$

$(P|A, ¾\%, 252\text{-}12) = 111.145$

$(P|A, ¾\%, 251\text{-}12) = 110.979$

$(P|A, i, m) = [1-(1+i)^{-m}]/i$



- \$804.62 until 252 too much, until 251 too little
- 251 notes of \$804.62, then pay off balance

B_{251} = \$124.81 = 89,316.48($F|P$, ¾%, 251 -12)
– 804.62($F|A$, ¾%, 251-12)

$Note_{252}$ = \$125.75 = 124.81($F|P$, ¾%, 252-251)

Step 3

- Eliminate 108 (360 - 252) notes of $804.62
- Invest savings at 1/2% / mo until t = 360

$116,015 = (804.62-125.75)($F|P$, ½%, 360-252)
+ 804.62 ($F|A$, ½%, 360-252)

- $116,015 more @ 360 than $10,000 on car
- If invest $10,000 @ 12 at 1/2% instead of paying off 3/4% loan

$56,727= $10,000($F|P$, ½%, 360-12)

- Eliminate debt @ 3/4% – Invest @ 1/2%

$59,288 = 116,015 - 56,727

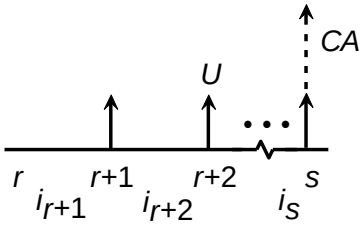
☺

- Yeah, the course is a pain, but ...

5.6 Multiple Interest Rates

- Same concepts as single rates
- Might not be able to use easy formulas
- Simplifications for constant regions

Compound Amounts

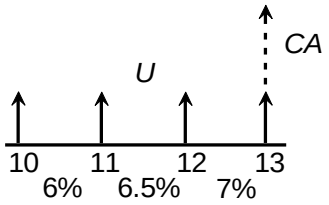


$$CA = U\gamma_{r+2}\gamma_{r+3}\cdots\gamma_s + U\gamma_{r+3}\cdots\gamma_s + \cdots + U$$

$$\Rightarrow CA = U\,(\gamma_{r+2}\gamma_{r+3}\cdots\gamma_s + \gamma_{r+3}\cdots\gamma_s + \cdots + 1)$$

$$\Rightarrow U = CA\,/\,(\gamma_{r+2}\gamma_{r+3}\cdots\gamma_s + \gamma_{r+3}\cdots\gamma_s + \cdots + 1)$$

Example 5.16 Multiple Rate Series and CAs



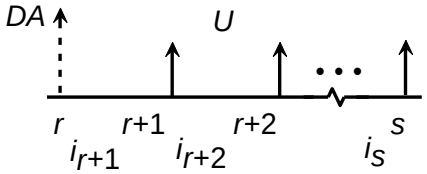
- U = $1,000, CA?

$4,417.47 = 1,000 [(1.06)(1.065)(1.07)
+ (1.065)(1.07) + 1.07 + 1]

- CA = 2,000, U?

$452.74 = 2,000 / [(1.06)(1.065)(1.07)
+ (1.065)(1.07) + 1.07 + 1]

Discounted Amounts

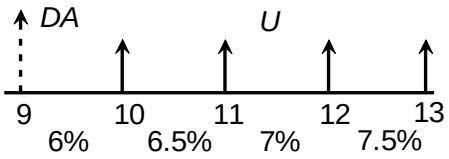


$$DA = U(\gamma_{r+1})^{-1} + \cdots + U(\gamma_{r+1}\gamma_{r+2}\cdots\gamma_s)^{-1}$$

$$\Rightarrow DA = U\,[\,(\gamma_{r+1})^{-1} + \cdots + (\gamma_{r+1}\gamma_{r+2}\cdots\gamma_s)^{-1}\,]$$

$$\Rightarrow U = DA\,/\,[(\gamma_{r+1})^{-1} + \cdots + (\gamma_{r+1}\gamma_{r+2}\cdots\gamma_s)^{-1}\,]$$

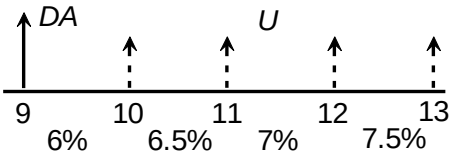
Example 5.17 Multiple Rate Series and DA



❖ U = \$1,000, DA?

$$\$3{,}427.19 = 1{,}000[(1.06)^{-1} + (1.06\times1.065)^{-1} + (1.06\times1.065\times1.07)^{-1} + (1.06\times1.065\times1.07\times1.075)^{-1}]$$

Step 2



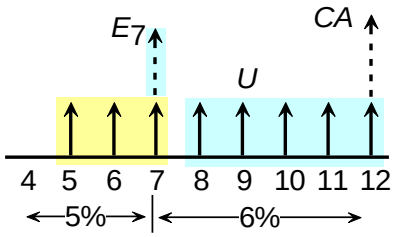
❖ DA = \$2,000, U?

$$\$583.56 = 2{,}000 / [(1.06)^{-1} + (1.06\times1.065)^{-1} + (1.06\times1.065\times1.07)^{-1} + (1.06\times1.065\times1.07\times1.075)^{-1}]$$

Regions with Constant Rates

- ❖ Discount and compound up to boundaries
- ❖ Equation relating series and DA or CA
- ❖ Solve equation for unknown

Example 5.18 Series CA with Regions



$$E_7 = U(F|A, 5\%, 7\text{-}4)$$

$$CA = E_7(F|P, 6\%, 12\text{-}7) + U(F|A, 6\%, 12\text{-}7)$$

$$CA = 9.8558U$$

Step 2

$$CA = 9.8558U$$

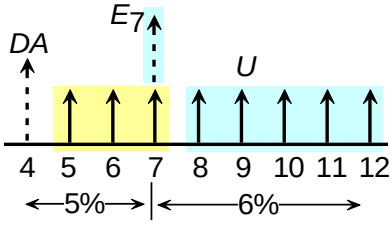
❖ If U equals \$1,000, then

$$CA = \$9{,}855.80 = 9.8558 \times 1{,}000$$

❖ If CA equals \$20,000, then

$$U = \$2{,}029.26 = 20{,}000 / 9.8558$$

Example 5.19 Series DA with Regions



$$E_7 = U(P|A, 6\%, 12\text{-}7)$$

$$DA = E_7(P|F, 5\%, 7\text{-}4) + U(P|A, 5\%, 7\text{-}4)$$

$$DA = 6.3620U$$

Step 2 $DA = 6.3620U$ ❖ If U equals \$1,000, then $DA = \$6{,}362.00 = 6.3620 \times 1{,}000$ ❖ If DA equals \$20,000, then $U = \$3{,}143.67 = 20{,}000 / 6.3620$	*Au revoir* Making money is rough work!