7: Compounding Frequency

Basics

- ❖ Compounding frequency affects rate of growth of savings or debt
  - ➢ \$100 after 1 year at 18% per year compounded annually ⇒ \$118.00
  - ➢ \$100 after 1 year at 18% per year compounded monthly ⇒ \$119.56

7.1 Nominal and Effective Interest

Lingua Franca (Language of the Trade)

- ❖ Consider CA of \$100 after one year
- ❖ "18% per year compounded yearly"
  - ➢ Means 18% per year
  - ➢ Compound once per yr @ 18%
  - ➢ CA = \$118.00 = 100(*F*|*P*, 18%, 1-0)
- ❖ "18% per year compounded monthly"
  - ➢ Means 1½% per month
  - ➢ Compound once per month @ 1.5%
  - ➢ CA = \$119.56 = 100(*F*|*P*, 1.5%, 12-0)

Nominal and Effective

- ❖ Interest rates equal the percentage growth in a CA over a stated period of time
  - ➢ "18% per year compounded monthly"
  - ➢ 19.56% = (119.56 – 100) / 100
- ❖ **Effective** or **actual** interest rate or **yield** is 19.56% ***per year***
- ❖ *Only* use of 18% is to compute 1.5 % per mo
  - ➢ **Nominal** (in name only) rate is 18% per yr

Observation

- ❖ Do not have to compound for an entire year for effective ***rate*** to be 19.56% per year
  - ➢ Four minute miler runs at a rate of 15 mph, even if does not run for an entire hour
- ❖ Annual Percentage Rate or **APR** is an approximate effective rate
  - ➢ Disclosure legally required
  - ➢ Disclosed rate can be very inaccurate

Rate Per Compounding Period Formula

- Rate per compounding period
  - r = nominal rate over a long period (year)
  - P = number of compounding periods (months) in the long period
  - $i_P = r / P$ ⇐
- "18% per year compounded monthly"
  - 1.5% (18%/12) is rate per compounding period
  - True rate that can be used in factors
    - Time measured in compounding periods

Effective Interest Formulas

- Effective rate for long period (year)
  - X = initial amount of savings or debt
  - $X(1+i_P)^P$ = CA after P short periods (months) in 1 long period (year)
- Long period % growth or effective interest
  - $i_e = [X(1+i_P)^P - X] / X$
    $= (1+i_P)^P - 1 = (1+r/P)^P - 1$ ⇐
- i_e known and i_P unknown
  - $i_P = (1+i_e)^{1/P} - 1$ ⇐

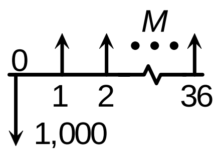
Observations

- Compound interest rates extrapolate *exponentially*, not linearly
- i_e = % growth per long period (year)
  - Can be used in factors
    - When time measured in long periods
- i_p = % growth per compounding period (month)
  - Can be used in factors
    - When time measured in comp periods
- Diagram and rate *must* be consistent!

Effect of Compounding Frequency for i_N=18%

Per	P	i_P	i_e
Yr	1	18.000%	18.000% = $(1+0.18/1)^1-1$
H-Yr	2	9.000%	18.810% = $(1+0.18/2)^2-1$
Qr	4	4.500%	19.252% = $(1+0.18/4)^4-1$
Mo	12	1.500%	19.562% = $(1+0.18/12)^{12}-1$
Wk	52	0.346%	19.685% = $(1+0.18/52)^{52}-1$
Day	365	0.049%	19.716% = $(1+0.18/365)^{365}-1$

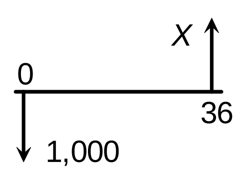
Example 7.2 Using Rates in Factors



i = 1% / mo

- Borrow $1,000
- 12% / yr compounded monthly
  - Interest is 1% per month
- Pay M / mo for 3 years
  - M = $33.21 = 1,000 (A | P, 1%, 36-0)
- ***Both*** cash flow diagram and interest rate expressed in terms of months

Single Payment – Months



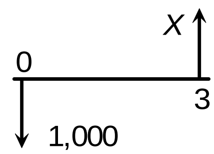
i = 1% / mo

- Borrow $1,000
  - 12% per year
  - Compounded monthly
  - Pay X in 3 years
- If use 1% per month, then draw cash flow diagram in terms of months

X = $1,430.77 = 1,000 ($F$ | P, 1%, 36-0)

Single Payment – Years

- If draw diagram in terms of years, must use yearly interest rate



$i = 12.683\% / yr$

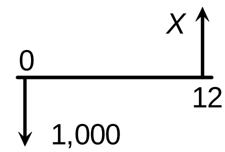
$i_{yr} = (1 + 0.12 / 12)^{12} - 1$

$= 12.683\%$

$X = \$1,430.77 = 1,000\ (F|P,\ 12.683\%,\ 3\text{-}0)$

Single Payment – Quarters

- If draw diagram in terms of quarters, must use quarterly interest rate



$i = 3.0301\% / qr$

$i_{qr} = (1 + 0.12 / 12)^{3} - 1$

$= 3.0301\%$

$X = \$1,430.77 = 1,000\ (F|P,\ 3.0301\%,\ 12\text{-}0)$

In general, ...

- Effective interest rate for any period may be used, as long as *same* period used for cash flow diagram
  - Eff rate also known as actual rate or yield

7.2 Loans and Unknown Interest Rates

Basics

- Stated rates for loans
  - Sometimes nominal
  - Might be based on profits for lender without including administrative or risk related (e.g., insurance) costs
- Borrower usually interested in rate that includes all costs

Different Viewpoints

- "Borrow" $1,000 and repay $1,100 in 1 year
  - Immediately pay $10 for loan processing costs and $25 for a credit check
  - Lender might state rate as 10%
  - Borrower actually receives

    $965 = 1,000 – 10 – 25

  - Rate from borrower's viewpoint

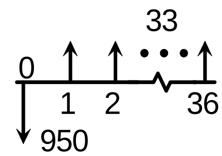
    13.99% = (1,100 – 965) / 965

Caveat Emptor

- A prudent borrower
  - Examines offerings of several lenders
  - Seeks out all costs
  - When possible, obtains written statements
  - Ignores stated rate and computes rate from his or her viewpoint that is based solely on cash flows

Example 7.3 Unknown Interest Rate

- "Borrow" $1,000
  - Credit check and administrative costs are $50 at time 0
  - Note on $1,000 is $32 / mo for 36 mos
  - Loan insurance of $1 per month required
- Cash flows
  - Time 0
    - Get $950 = 1,000 – 50
  - Times 1 to 36
    - Pay $33 = 32 + 1



Borrower's Viewpoint

- Use trial-and-error to solve the discounted amount equation for the unknown monthly interest rate, 1.262%:

$$\$950 = 33\,(P|A,\, i_m\,,\, 36\text{-}0)$$

$\Rightarrow$ $(P|A,\, i_m,\, 36\text{-}0) = 29.0606 = 950/33$

  - Use formula and calculator's root solver
  - Search table and interpolate

$$i_m = 1.262\%$$

- Yearly rate from borrower's viewpoint

$$16.243\% = (1 + 0.01262)^{12} - 1$$

Possible Lender's Viewpoint

- If "borrow" $1000 and "pay" $32 / mo

$$\$1{,}000 = 32\,(P|A,\, i_m\,,\, 36\text{-}0)$$

$$i_m = 0.786\%$$

$$i_N = 9.429\% = 12 \times 0.786\%$$

- Possible quote: 9.429% compounded monthly
- Really nothing gained by trying to determining how lender's quote computed

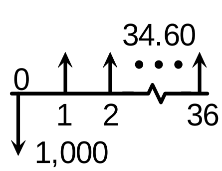
Example 7.4 Selecting the Best Loan

- Borrow $1,000 as before
- This time, finance credit check and administrative costs of $50 at 9.429% per year compounded monthly: $1.60 / mo
- Note on $1,000 is $32 / mo for 36 months
- Insurance of $1 / mo required
- Solve for i_m

$1000 = 34.60(P|A,\, i_m,\, 36\text{-}0)$

$(P|A,\, i_m,\, 36\text{-}0) = 28.9017$

$i_m = 1.239\%$



Selecting the Best Loan – Step 2

- Yearly rate from borrower's viewpoint

$$15.923\% = (1 + 0.01239)^{12} - 1$$

- This loan "looks better" than former one, since its rate is 15.923% instead of 16.243%
- However, objective = max final balance
- Suppose borrower saves @ 0.5% / mo
- Current loan leaves extra $50 (1,000 – 950) in savings at t = 0, but requires taking out extra $1.60 (34.60 – 33) each month

Selecting the Best Loan – Step 3

- CA of saving \$50 at $t = 0$ and paying \$1.60 at $t = 1, 2, \ldots, 36$

$$-\$3.10 = 50(F|P, 0.5\%, 36\text{-}0) - 1.60\,(F|A, 0.5\%, 36\text{-}0)$$

- Savings are \$3.10 smaller after 36 months if take second loan
- Lower rate, but borrowing more
- Best rate usually good loan, but does not guarantee best balance
  - Effect of current investment on others not considered. More in later chapters. ☺

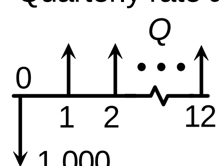
7.3 Multi-Period Series

Basics

- Equal, regular cash flows
  - Frequency a multiple of comp period
- Two methods for CAs and DAs
  - Use effective interest rate
  - Convert flows to US w/ A|F or A|P

Example 7.5 Monthly Rate, Quarterly Flows

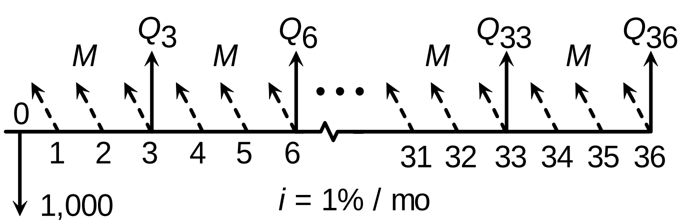
- Borrow \$1,000 at 12% per yr comp monthly
- Determine quarterly payments for 3 years
- Effective interest approach
- Quarterly rate *and* quarterly time scale



$$3.0301\% = (1 + .01)^3 - 1$$

$$\$100.64 = 1{,}000\,(A|P, 3.0301\%, 12\text{-}0)$$

Conversion to Uniform Series



- If monthly payments @ 1% made to lender

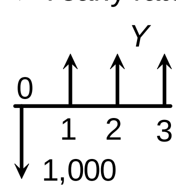
$$M = \$33.21 = 1{,}000(A|P, 1\%, 36\text{-}0)$$

- Instead, pay CA of each quarter's three monthly flows

$$Q = Q_3 = \cdots = Q_{36} = \$100.64 = 33.21(F|A, 1\%, 3)$$

Example 7.5 Monthly Rate and Yearly Flows

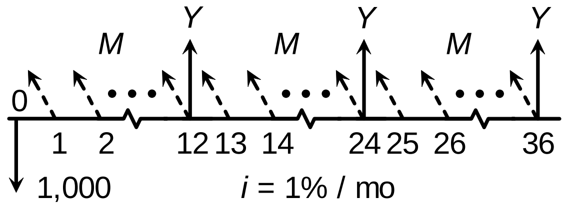
- Borrow \$1,000 at 12% per yr comp monthly
- Determine yearly payments for 3 years
- Effective interest approach
- Yearly rate *and* yearly time scale



$$12.6825\% = (1 + .01)^{12} - 1$$

$$Y = \$421.24 = 1{,}000\,(A|P, 12.6825\%, 3\text{-}0)$$

Conversion to Uniform Series



- If monthly payments @ 1% made to lender

  $M = \$33.21 = 1{,}000(A|P, 1\%, 36\text{-}0)$

- Instead, pay CA of each year's twelve monthly flows

  $Y = \$421.24 = 33.21(F|A, 1\%, 12)$

Observations

- Make interest rate and time scale ***consistent***
- Same answer
  - Possible minor round-off differences
- Expected to know both ways
  - Using rates in tables
  - Not using rates in tables

7.4 Cash Flows Within Periods

Topics

- Two cases
  - Compounding occurs within the period
  - No compounding within the period

Compounding Within the Period

- Example: yearly effective rate known, but compounding and cash flows occur monthly
- Compute rate for shorter period and solve problem on that basis

  $$i_P = (1 + i_e)^{1/P} - 1$$

- Yearly effective rate of 9.381% with monthly cash flows and compounding

  $$i_m = 0.75\% = (1 + 0.09381)^{1/12} - 1$$

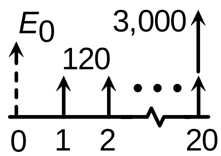
- Draw cash flow diagram on a monthly basis and use 0.75%

No Compounding Within Period

- Example: rate and compounding are yearly
- Deposit at month 4 does not begin earning interest until start of next year
- Withdrawal at month 4 reduces balance at beginning of year on which interest is paid
- Clearly harms consumer if compounding is yearly, less harmful for more common daily compounding

Example 7.7 Bonds

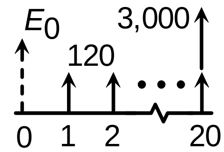
- Investor wants to earn 10% per year
- Bond: redemption value of $3,000 in 10 years
  - Coupon rate is 8% payable semiannually
  - Most to pay for bond?
- "8% payable semi-annually" means 4% per half-year
  - 20 coupons of $120 = 4% × $3,000



Bonds - Step 2

- Rate investor must earn semi-annually to earn 10% per year

  $i_S = 4.881\% = (1 + 0.10)^{1/2} - 1$



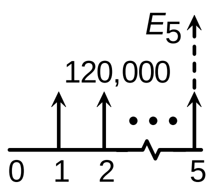
- Maximum to pay is discounted amount at 4.881% per half-year

  $E_0 = 120(P|A, 4.881\%, 20\text{-}0) + 3{,}000(P|F, 4.881\%, 20\text{-}0)$

  $E_0 = \$2{,}667.32$

Ex. 7.8 Not Compounding Within a Period

- Deposit sales of $10,000 per month for 5 years
  - Effective interest is 9% / yr with monthly compounding
- Determine CA as if no compounding within the year
  - Figure's $120,000 / yr ignores compounding



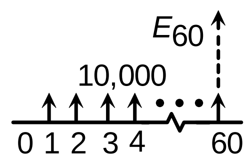
$\$718{,}165.27 = 120{,}000\ (F|A, 9\%, 5\text{-}0)$

Recognizing Compounding

- Figure shows monthly flows
- Monthly rate is

  $i_m = (1 + 0.09)^{1/12} - 1$

  $i_m = 0.7207\%$



$\$747{,}328.69 = 10{,}000\ (F|A, 0.7207\%, 60\text{-}0)$

- Previous erroneous CA is 3.9% too small
  - Erroneous CA becomes 6.3% too small if effectively yearly rate is 15%

Observations

- If errors of this magnitude can affect a decision, then use modern computational aids like spreadsheets to model as accurately as possible
  - Example problems and homework typically assume end-of-year flows for simplicity in illustrating concepts

7.5 Continuous Compounding

Common Occurrence

- Some financial institutions offer continuous compounding
- Continuous reinvestment common in industry

Different Compounding Periods for $i_N = 18\%/Yr$

Compounding Periods	
Period	i_e
Year	18.000%
Half-Year	18.810%
Quarter	19.252%
Month	19.562%
Week	19.685%
Day	19.716%

- Notice limiting behavior as compounding frequency increases

Nominal and Effective Rates

- For a given nominal rate r, the rate per compounding period is

$$i_P = r/P$$

  - Rate per period decreases as number of periods increase
- Eventually, a limiting condition occurs

$$i_e = \lim_{P \to \infty} (1 + r/P)^P - 1$$

$$i_e = e^r - 1$$

  - $effective = e^{nominal} - 1$
  - r and i_e are both for the same time period

Illustration

- i_N = 18% / yr compounded continuously

$$\Rightarrow \; i_e = 19.722\% = e^{0.18} - 1$$

  - 19.722% is effective yearly rate
  - Use 19.722% in factors with yearly flows
- If only effective rate known, $i_e = e^r - 1$ implies
  - $r = ln(1 + i_e)$
  - $nominal = \ln(1 + effective)$
- i_e = 19.722% / yr w/ continuous compounding

$$\Rightarrow \; i_N = 18\% = ln(1 + 0.19722)$$

  - 18% is nominal yearly rate

Nominal Rates Extrapolate Linearly

- Nominal rates ignore compounding
- Is 12% / yr compounded monthly the same rate as 3% / qr compounded monthly?
  - Both mean exactly the same thing
    - 1% / mo with monthly compounding
- As long as nominal period larger than compounding period, meaning is same
- Continuous compounding not different
  - Just has very short compounding periods
- 12% / yr compounded continuously same as 3% / qr compounded continuously

Then there's math

- If r is yearly nominal rate w/ cont compounding
  - $i_{yr} = e^r - 1$
- If P periods per year, then
  - $i_P = (1 + i_{yr})^{1/P} - 1$
  - $i_P = (1 + e^r - 1)^{1/P} - 1$
  - $i_P = e^{r/P} - 1$
- i_P is true or effective rate per period
  - $effective = e^{nominal} - 1$
  - r/P is nominal rate per period with continuous compounding

Linear Extrapolations

- 12% / yr compounded continuously same as
  - 6% / half yr compounded continuously
  - 3% / qr compounded continuously
  - 1% / mo compounded continuously
  - 24% biennially compounded continuously
- *Nominal* rates for periods longer than the compounding period extrapolate linearly
  - Any nominal rate for continuous compounding exptrapolates linearly
  - Continuous has shortest possible compounding periods

Synchronizing Flows and Effective Rates

- 18% per year compounded continuously
- For monthly flows, *nominal* rate is 1.5% (18%/12) per month compounded continuously
  - $i_m = 1.511\% = e^{0.18/12} - 1$
- Always compute effective rates so they and time periods of flows are consistent

Example 7.9 Rates for Different Periods

Effective Rates for $i_N = 0.12$ / Year		
Period	Nom	Rate per Period
Month	0.12/12	$i_m = 1.0050\% = e^{0.01} - 1$
Quarter	0.12/4	$i_q = 3.0455\% = e^{0.03} - 1$
Half yr	0.12/2	$i_h = 6.1837\% = e^{0.06} - 1$
Year	0.12	$i_y = 12.7497\% = e^{0.12} - 1$
Two yr	0.12x2	$i_b = 27.1249\% = e^{0.24} - 1$

Factors

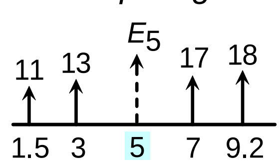
- Effective rates accurately measure rate of growth of savings or debt for each period, so they can be used in factors
  - 12% per year compounded continuously
  - Discounted amount of $1000 occurring 8 months from now

    $923.12 = 1,000(*P*|*F*, 1.0050%, 8-0)
- Must evaluate factor using calculator

Cont Compounding Single Payment Factors

- $e^r - 1$ is eff rate corresponding to nom rate r
- $(P|F, i_e, m) = (1+e^r-1)^{-m} = e^{-rm}$
  - Use e^{-rm} instead of $(P|F, r = x\%, m)$
- $(F|P, i_e, m) = (1+e^r-1)^{m} = e^{rm}$
  - Use e^{rm} instead of $(F|P, r = x\%, m)$
- r and m must be dimensionally consistent, such as both being yearly values or both being monthly values
- For other factors, use the effective rate i_e in formulas

Ex 7.10 Cont Comp Single Payment Factors



- Interest is 12% / yr compounded continuously
- Cash flows are monthly
  - Time measurement must be *consistent*
  - Monthly nominal rate is 1% = 12% / 12

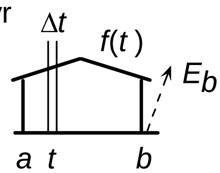
$$E_5 = 11e^{0.01(5-1.5)} + 13e^{0.01(5-3)} + 17e^{-0.01(7-5)} + 18e^{-0.01(9.2-5)} = \$58.58$$

7.6 Continuous Cash Flows

Common Occurrence

- Occur frequently in practice
  - ATM machines, on-line credit card gasoline purchases
  - Industrial maintenance costs during a year

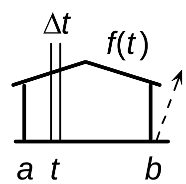
Compound Amount



- Funds flow rate $f(t)$, \$10,000 / yr
- Flow in $\Delta t \approx f(t)\Delta t$
- Flow in day is 10,000(1/365)
  - Dimensional consistency
- CA of small flow is $f(t)\Delta t e^{r(b-t)}$
  - t, $f(t)$, r, a, and b dimensionally consistent
- Sum all compound amounts between a and b

$$E_b = \int_a^b f(t) e^{r(b-t)} dt$$

Constant Funds Flow

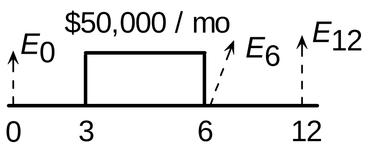


$$\mathbf{E_b} = \int_a^b f(t) e^{r(b-t)} dt$$

- Algebraic formulas depend on $f(t)$
  - Most common case is $f(t) = K$, such as \$10,000 per year
  - Above integral simplifies to

$$E_b = K \frac{e^{r(b-a)} - 1}{r}$$

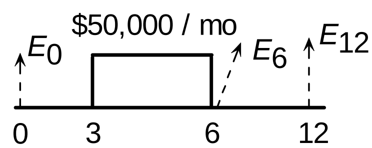
Example 7.11 Constant Funds Flow Function



- On-line credit card purchase of gasoline
- Interest is 6% / yr, compounded continuously
- Revenues are \$50,000 / mo from mo 3 to 6
- CA at times 6 and 12? DA at time 0?
- Time scale
  - Months: nominal rate / mo = 0.5% = 6% / 12
  - Years: nominal rate / yr = 6% and change times to 0.25 (3), 0.5 (6), and 1.0 (12)

Step 2

r = 0.5% / mo



$$E_6 = \$151{,}130.65 = 50{,}000 \frac{e^{0.005(6-3)} - 1}{0.005}$$

$$E_0 = \$146{,}664.06 = E_6 e^{-0.005(6-0)}$$

$$E_{12} = \$155{,}733.26 = E_6 e^{0.005(12-6)}$$

Example 7.12 Discrete Approximations

- Sometimes treat continuous flows as if they occurred at end of period
  - Continuous flow of $365,000 per year
  - i_e = 10% / yr with continuous compounding
- CA at end of year?
  - Nom rate / yr with continuous compounding

$$r = 9.531\% = ln(1.10)$$

$$CA = \$382{,}960.14 = 365{,}000 \frac{e^{0.09531(1-0)} - 1}{0.09531}$$

- Abs % Err = 4.690%

  = |365,000 − 382,960.14| / 382,960.14

Effect of Treating as If End-of-Year Flows

Discrete Approximations		
Period	CA	% Error
Actual	$382,960.14	0.000%
Day	382,910.15	0.013
Week	382,592.65	0.096
Month	381,374.37	0.414
Quarter	378,254.04	1.229
Half-Yr	373,697.05	2.419
Year	365,000.00	4.690

Absolute Percent Errors

- Daily flows
  - $/Day = $1,000 = 365,000 / 365
  - $i_d = 0.02612\% = e^{0.09531/365} - 1$
  - $382,910.15 = 1,000($F|A$, 0.02612%, 365-0)
  - $0.013\% = \frac{|382{,}910.15 - 382{,}960.14|}{382{,}960.14}$

Observations

- Errors get worse for higher interest rates
- In general, fairly little error with monthly flows
  - Assume end-of-year flows for most of course
  - Make basic points more quickly
  - Reduce homework time
- At work, determine acceptable level of error