Partial groups and higher Segal conditions

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(joint work with Philip Hackney)

In Chermak's original group theoretic formulation [3], a partial group is a set together with a multivariable product that is only defined on a subset of multipliable words in the underlying set, together with an inversion for that product. Here we report on work understanding the higher Segal conditions of Dyckerhoff and Kapranov [5] in the context of partial groups. We define a new invariant of a partial group, its *degree*, develop the discrete geometry of partial group actions as a tool for computing this invariant, and through this we show that partial groups form a rich class of higher Segal sets of finite group theoretic significance.

Partial groups as symmetric sets. Partial groups seem to be best regarded as certain types of symmetric simplicial sets [9]. The simplex category Δ has objects the totally ordered sets $[n] = \{0, 1, \ldots, n\}$ $(n \geq 0)$ and morphisms monotone maps $[m] \to [n]$. The symmetric simplex category $\Upsilon \supset \Delta$ has the same objects, but has morphisms all functions. A symmetric simplicial set is a presheaf on Υ, i.e., a functor $\Upsilon^{\mathrm{op}} \to \mathsf{Set}$. It amounts to a simplicial set X together with compatible actions of the symmetric groups $\Sigma_{[n]}$ on the sets $X_n := X([n])$ of n-simplices for each n.

The standard example of a simplicial set is the nerve BC of a category. If the category is a groupoid G, then its nerve BG enjoys the structure of a symmetric simplicial set. Grothendieck's Nerve Theorem characterizes nerves of categories (resp. groupoids) as those simplicial sets (resp. symmetric sets) X for which the Segal maps

$$\mathscr{E}_n \colon X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are bijections for all $n \geq 2$, a stipulation on X called the *Segal condition*. The map $\mathscr{E}_n$ sends an n-simplex x to its standard spine $(\epsilon_{0,1}^* x, \ldots, \epsilon_{n-1,n}^* x)$, where $\epsilon_{i,j}$ is the map from $[1]$ to $[n]$ that sends 0 to i and 1 to j.

We consider two different weakenings of the Segal condition and their interaction. The first leads to one model for a *partial category*, but we will focus on the symmetric version, a model for *partial groupoid*.

Definition 1. A symmetric set is *spiny* if the Segal maps are all injections. A *partial groupoid* is a spiny symmetric set. A *partial group* is a reduced partial groupoid, i.e., one with a single 0-simplex.

Spininess, originally an insight of González [8], allows one to write an n-simplex f in a partial groupoid as $f = [f_1|\cdots|f_n]$ where $f_i = \epsilon_{i-1,i}^* f$ is the i-th principal edge. Chermak's partial product is given by the span $X_1 \times_{X_0} \cdots \times_{X_0} X_1 \leftarrow X_n \to X_1$, where the right map is $\epsilon_{0,n}^*$. The inversion $f \mapsto f^{-1}$ is the action of the longest element $i \mapsto n - i$ in $\Sigma_{[n]}$. Many partial groups of interest fall under the umbrella of the next example, but not all.

Example 1. Let G be a group and S a set equipped with a partial action of G in the sense of Exel [6] (see also [10]). This gives rise to a transporter groupoid

$T_S(G)$ having object set S and morphisms $s \xrightarrow{g} g \cdot s$ whenever g acts on s, as well as a functor $T_S(G) \to G$. Let $L_S(G) \subseteq BG$ be the image of the corresponding map on nerves. Then $L_S(G)$ is a partial group whose n-simplices are those $[g_1|\cdots|g_n]$ such that there is $s \in S$ with $g_1 \cdot s$ defined, $g_2 \cdot (g_1 \cdot s)$ defined, and so forth.

Higher Segal conditions. The second weakening is a family of associativity conditions on simplicial objects, the higher Segal conditions of Dyckerhoff and Kapranov [5]. They come in lower and upper versions, one for each $d \geq 1$; the ordinary Segal condition corresponds to lower 1-Segality. Satisfaction of either the lower or the upper $(d-1)$-Segal condition implies lower *and* upper d-Segality. The 2-Segal conditions were independently introduced by Gálvez-Carrillo–Kock–Tonks [7] and have been studied intensely in recent years. We explain, based on a distillation of [11], the meaning of the higher Segal conditions here only for partial groupoids X, where they become concrete and combinatorial, and where they collapse to the lower odd conditions. (For example a partial group is lower or upper 2-Segal if and only if it is lower 1-Segal, i.e., a group.)

If X is a partial groupoid, then X is *lower* $(2k-1)$*-Segal* if, given $n \geq 1$ and a *gapped sequence*

$$0 \leq i_0 \ll i_1 \ll \cdots \ll i_k \leq n,$$

of length $k+1$ (i.e., adjacent terms are at distance at least two), and given a potentially multipliable word $w = (f_1, \ldots, f_n) \in X_1 \times_{X_0} \cdots \times_{X_0} X_1$ of length n,

$$d_{i_0} w,\ d_{i_1} w,\ \ldots,\ d_{i_k} w \in X_{n-1} \implies w \in X_n.$$

Here, if $i_0 = 2$ say, the expression $d_2 w$ carries the tacit assumption that $[f_2|f_3] \in X_2$, so that one can form the word $d_2 w = (f_1, d_1[f_2|f_3], f_4, \ldots, f_n)$.

Definition 2. The *degree* $\deg(X)$ of a partial groupoid X is the smallest k such that X is lower $(2k-1)$-Segal.

Example 2. If G is a group, the classifying space for commutativity $B_{\mathrm{com}}(G) \subseteq BG$ is the partial group with n-simplices $[g_1|\cdots|g_n] \in BG_n$ whenever $g_1, \ldots, g_n$ pairwise commute, cf. [1]. It is lower 3-Segal. The first of the lower 3-Segal conditions ($n = 4$) says that if $g_1, \ldots, g_4$ are elements of G such that g_1, g_2, g_3 commute, g_1, g_2g_3, g_4 commute, and g_1, g_2, g_3 commute, then all four elements commute. $B_{\mathrm{com}}(G)$ is lower 1-Segal only if G is abelian, in which case it is BG. So $\deg(B_{\mathrm{com}}(G)) = 1$ or 2 depending on whether G is or is not abelian.

Degree as Helly number. We now describe tools for computing the degree of a partial group. As motivation, we start with the following subexample of Example 1. Let Φ be a finite root system. A fixed set Φ^+ of positive roots admits a partial action by the Weyl group W, and we form the partial group $L = L_{\Phi^+}(W)$. It has elements/1-simplices $L_1 = W \backslash \{w_0\}$, where w_0 is the longest element.

Theorem 1 (Hackney–L.). *The degree of $L_{\Phi^+}(W)$ is the Helly number of Φ^+ with respect to convex subsets.*

The classical Helly number is the smallest h such that whenever each collection of h members of a family of convex sets has nonempty intersection, then the entire family has nonempty intersection. Helly's Theorem from 1913 says that the Helly number for convex sets in $\mathbb{R}^d$ is $d+1$. In the situation of the theorem, a subset A of Φ^+ is convex if it coincides with its convex cone $\mathbb{R}_{\geq 0}A \cap \Phi^+$. We compute the Helly number of Φ^+ explicitly by showing that it is closely related to the maximal dimension of an abelian subalgebra of the associated semisimple Lie algebra, as was computed by Malcev. In fact it coincides with this number in simply laced types. For example, $\deg(L_{\Phi^+}(W)) = 16, 27, 36$ in cases E_6, E_7, E_8. In other words, $L_{\Phi^+}(W)$ is 71-Segal when $W = W(E_8)$, but not 69-Segal.

It turns out the above story persists for arbitrary partial groups in almost full generality.

Definition 3. An *action* of a partial groupoid L is a map $\rho\colon E \to L$ of partial groupoids with the following partial lifting condition: for each n-simplex $g \in L_n$ and each $x \in E_0$ mapping to the source of g, there is at most one $e \in E_n$ such that $\rho(e) = f$.

We interpret this through a Grothendieck correspondence for partial actions, giving a way for a simplex g of L to act on an element $x \in E_0$, by choosing a lift e of g with source x and setting $g \cdot x = y$, the target of e. The *domain* $\mathrm{D}_\rho(g)$ of g is the set of x's for which there is such a lift. This is of course modeled on the map $BT_S(G) \xrightarrow{\rho} L_S(G) \subseteq BG$ of Example 1 and on domains of partial maps. In that case, $E = BT_S(G)$ is (the nerve of) a groupoid, and ρ is surjective, a situation we will temporarily call "nice" here.

Any action gives rise to a closure operator cl on E_0, sending a subset A of E_0 to the intersection of those domains of simplicies that contain A. An indexed family of subsets $\{A_1, \ldots, A_k\}$ of such a closure space is *Helly independent* if

$$\bigcap_{i=1}^{k} \mathrm{cl}\left(\bigcup_{j\neq i} A_j\right) = \varnothing,$$

and the *Helly number* $h(\rho)$ is the maximal size of an independent set. (This is an equivalent definition, dual to the one given before.)

Theorem 2 (Hackney–L.)**.** *If L is a partial group that is not a group and ρ is a nice action of L, then* $\deg(L) \leq h(\rho)$. *Also,* $h(\rho) \leq \deg(L)$ *if domains of simplices satisfy the descending chain condition.*

Every partial group has an explicit, nice action, so Theorem 2 gives effective means for computing the degree. It applies, for example, to the discrete localities [4] associated with p-local compact groups [2].

References

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