Approximate one-sided tolerance limits in random effects model and in some mixed models and comparisons

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Approximate one-sided tolerance limits in random effects model
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An approximate closed-form one-sided tolerance limit (TL) in a general mixed model is proposed. One-sided TLs for the distribution of observable random variable and for the distribution of unobservable random variable in one-way random model are obtained as special cases from the one for the general mixed model. Applications to a two-way nested random model are also given. The merits of the TLs are evaluated using Monte Carlo simulation and compared with the existing ones. Our comparison studies indicate that the approximate TLs are quite satisfactory for all parameter and sample size configurations, and better than the existing ones in some cases. Approximate confidence intervals for exceedance probabilities in one-way random effects model are also proposed. The procedures are illustrated using three examples.

\textbf{Keywords:} exceedance probability; generalized variable; Graybill–Wang approximation; MOVER; Satterthwaite approximation; nested model; survival probability; variance components

1. Introduction

Tolerance intervals (TIs) are widely used in many areas of sciences including engineering, quality assessment, pharmaceutical industry. Apart from these areas of sciences, TIs are also used to assess the exposure level in a workplace or pollution level in an environment.[1,2] The problems of constructing one-sided tolerance limits (TLs) and two-sided TIs in one-way random models and in some general mixed models have been investigated by several authors for the past three decades. For example, see Mee and Owen,[3] Beckman and Tietjen,[4] Bhaumik and Kulkarni,[5] Vangel,[6] Liao and Iyer,[7] Krishnamoorthy and Mathew,[8] and Liao et al.,[9] and the references therein. So far all available methods are approximate, and no single approach is satisfactory for all parameter–sample size configurations.

Krishnamoorthy and Mathew [8] have proposed one-sided TLs based on the generalized variable approach, and an approximate one based on the noncentral $t$ percentiles for one-way random model with balanced or unbalanced data. More specifically, they proposed approximate TLs for the distribution of observable random variable and for the distribution of unobservable ‘true value’. Monte Carlo evaluation and comparison studies by these authors indicated that none of the methods, including Vangel’s [6] methods, is satisfactory for all situations. The approximate
TLs based on the noncentral $t$ percentiles are simple to compute and are satisfactory when the intraclass correlation $\rho \geq 0.5$, which maybe a realistic condition in many applications. In some cases, the generalized TLs \[8,9\] and Vangel’s \[6\] TLs could be very conservative, yielding TIs that are unnecessarily wide. Recently, Hoffman \[10\] has proposed an approximate closed-form TLs for one-way random models which are quite conservative; a bootstrap-based adjustment was suggested to improve the TLs, and the improved ones are comparable with the generalized TLs by Krishnamoorthy and Mathew.\[8\] An excellent review of available procedures for constructing one-sided as well as two-sided TLs in one-way random models and in general mixed models, and applications of TLs, see Chapters 4–6 of the book by Krishnamoorthy and Mathew.\[11\]

In this article, we shall provide simple closed-form approximate one-sided TLs in a general mixed model using the method of variance estimate recovery (MOVER) by Zou and Donner \[12\] and Zou et al.\[13,14\] In the context of ‘Bland-Altman limits of agreement’, Zou \[15\] has used the MOVER to develop confidence intervals (CIs) for quantiles in a one-way random model. Noting that a one-sided TL is a one-sided confidence limit (CL) for an appropriate quantile, Zou’s result can be used to find approximate one-sided TLs for the distribution of observable random variable in a one-way random model. We here generalize Zou’s result by obtaining one-sided TLs in a general mixed model, and show applications in finding TLs for the unobservable random variable in one-way random models, and for finding TLs for observable as well as for unobservable random variables in a two-way nested model with random effects. Furthermore, we shall compare the approximate TLs with the popular ones proposed in the literature, and the generalized TLs proposed by Fonseca et al. \[16\] for a two-way nested model. A related problem of estimating the exceedance probability (survival probability) in a one-way random model is also addressed and an approximate solution is provided.

The rest of the article is organized as follows. In the following section, we describe the MOVER approach for finding CLs for a linear combination of parameters, and for finding TLs in a general mixed model. The TLs in one-way random model with balanced or unbalanced data are described, and their coverage properties are studied in Section 3. Also, a closed-form solutions to the related problem of finding CLs for the survival probability (or exceedance probability) are given. Approximate TLs in a two-way nested model with random effects are given in Section 4, and they are compared with the generalized TLs given in Fonseca et al.\[16\] The results are illustrated using three examples in Section 5, and some concluding remarks are given in Section 6.

### 2. MOVER TLs in general mixed model

The MOVER CI by Zou and Donner,\[12\] Zou et al. \[13,14\] for a linear combination of parameters $\theta_1, \ldots, \theta_g$ is described as follows. Let $\hat{\theta}_i$ be an unbiased estimate of $\theta_i$, $i = 1, \ldots, k$. Assume that $\hat{\theta}_1, \ldots, \hat{\theta}_g$ are independent. Further, let $(l_i, u_i)$ denote the $1 - \alpha$ CI for $\theta_i$, $i = 1, \ldots, k$. The $1 - \alpha$ MOVER CI $(L, U)$ for $\sum_{i=1}^{g} c_i \hat{\theta}_i$ can be expressed as

$$L = \sum_{i=1}^{g} c_i \hat{\theta}_i - \sqrt{\sum_{i=1}^{g} c_i^2 (\hat{\theta}_i - l_i^*)^2}$$

with $l_i^* = \begin{cases} l_i & \text{if } c_i > 0, \\ u_i & \text{if } c_i < 0 \end{cases}$ (1)

and

$$U = \sum_{i=1}^{g} c_i \hat{\theta}_i + \sqrt{\sum_{i=1}^{g} c_i^2 (\hat{\theta}_i - u_i^*)^2}$$

with $u_i^* = \begin{cases} u_i & \text{if } c_i > 0, \\ l_i & \text{if } c_i < 0. \end{cases}$ (2)
It should be noted that Graybill and Wang [17] first obtained the above CI for a linear combinations variance components. Zou and co-authors gave a different argument so as to the above CI is valid for any parameters.

Finding a \((p, 1 - \alpha)\) upper TL in one-way random model or in a general mixed model simplifies to finding a \(1 - \alpha\) upper CL for the \(p\) quantile of a \(N(\mu, \sum_{i=1}^{g} a_i \sigma_i^2)\) distribution, where \(a_i\)'s are known constants, and \(\sigma_i^2\)'s are linear combinations of variance components. Noting that the \(p\) quantile of the \(N(\mu, \sum_{i=1}^{g} a_i \sigma_i^2)\) distribution is given by \(\mu + z_p \sqrt{\sum_{i=1}^{g} a_i \sigma_i^2}\), an upper TL can be readily obtained using \(1 - \alpha\) upper CLs for \(\mu\) and \(\sqrt{\sum_{i=1}^{g} a_i \sigma_i^2}\). Let \(U_\mu\) and \(U_v\) denote \(1 - \alpha\) upper CLs for \(\mu\) and \(\sum_{i=1}^{g} a_i \sigma_i^2\), respectively. The \((p, 1 - \alpha)\) upper TL for the \(N(\mu, \sum_{i=1}^{g} a_i \sigma_i^2)\) distribution can be obtained from Equation (2) as

\[
\hat{\mu} + \left\{ z_p \left( \sum_{i=1}^{g} a_i \sigma_i^2 \right)^{1/2} + \left[ (U_\mu - \hat{\mu})^2 + z_p^2 \left( U_v - \left( \sum_{i=1}^{g} a_i \hat{\sigma}_i^2 \right)^{1/2} \right)^2 \right]^{1/2} \right\}, \tag{3}
\]

where \(x_+ = \max\{x, 0\}\), and \(\hat{\sigma}_i^2\)'s are independent unbiased estimates of \(\sigma_i^2\)'s. The \((p, 1 - \alpha)\) MOVER lower TL is similarly obtained from Equation (1) as

\[
\hat{\mu} - \left\{ z_p \left( \sum_{i=1}^{g} a_i \sigma_i^2 \right)^{1/2} + \left[ (L_\mu - \hat{\mu})^2 + z_p^2 \left( U_v - \left( \sum_{i=1}^{g} a_i \hat{\sigma}_i^2 \right)^{1/2} \right)^2 \right]^{1/2} \right\}, \tag{4}
\]

where \(L_\mu\) is a \(1 - \alpha\) lower CL for \(\mu\). The required upper CL \(U_v\) for \(\left( \sum_{i=1}^{g} a_i \hat{\sigma}_i^2 \right)^{1/2}\) can be obtained using the MOVER method. We also note that \((U_\mu - \hat{\mu})^2 = (L_\mu - \hat{\mu})^2\) for all the cases considered in the sequel, as a result, the expressions within curly parentheses of Equations (3) and (4) will be the same.

One-sided TLs for various problems involving random models with balanced or unbalanced data can be easily constructed in a straightforward manner using the above general form. In fact, for each problem, we simply need to identify the coefficients \(a_i\) and the individual CLs for \(\mu\), and \(\sigma_i^2\), \(i = 1, \ldots, g\).

3. One-way random models

Let \(Y_{ij}\) denote the \(j\)th observation corresponding to the \(i\)th level, assumed to follow the one-way random model

\[
Y_{ij} = \mu + \tau_i + e_{ij}, \quad j = 1, 2, \ldots, n_i, \ i = 1, 2, \ldots, k, \tag{5}
\]

where \(\mu\) is an unknown general mean, \(\tau_i\)'s represent random effects, and \(e_{ij}\)'s represent error terms. It is assumed that \(\tau_i\)'s and \(e_{ij}\)'s are all mutually independent with \(\tau_i \sim N(0, \sigma^2)\) and \(e_{ij} \sim N(0, \sigma^2)\). Thus, the observable random variable \(Y_{ij} \sim N(\mu, \sigma^2 + \sigma^2)\), and the unobservable ‘true value’ associated with the \(i\)th level \(\mu + \tau_i \sim N(\mu, \sigma^2)\). We shall consider the problems of constructing one-sided TLs for the distribution \(N(\mu, \sigma^2 + \sigma^2)\), and for the distribution \(N(\mu, \sigma^2)\).
Define
\[
\tilde{n} = \frac{1}{k} \sum_{i=1}^{k} n_i^{-1}, \quad \tilde{Y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}, \quad \tilde{Y} = \frac{1}{k} \sum_{i=1}^{k} \tilde{Y}_i, \quad SS_y = \sum_{i=1}^{k} (\tilde{Y}_i - \tilde{Y})^2, \quad \text{and}
\]
\[
SS_e = \sum_{i=1}^{k} \sum_{j=1}^{m_i} (Y_{ij} - \tilde{Y}_i)^2.
\]

Then,
\[
\tilde{Y}_i \sim N\left(\mu, \sigma^2 + \frac{\sigma^2_e}{n_i}\right), \quad \tilde{Y} \sim N\left(\mu, \frac{\sigma^2 + \tilde{n}\sigma^2_e}{k}\right), \quad \text{and} \quad SS_e \sim \sigma^2 \chi^2_{N-k} \quad \text{with} \quad N = \sum_{i=1}^{k} n_i.
\]

It can be verified that \(E(SS_y) = (k-1)(\sigma^2 + \tilde{n}\sigma^2_e)\). Furthermore,
\[
\frac{SS_y}{\sigma^2 + \tilde{n}\sigma^2_e} \sim \chi^2_{k-1}.
\]

When \(n_1 = \cdots = n_k\), the above distributional result is exact, and \(\tilde{Y}, SS_y\) and \(SS_e\) are mutually independent. For the unbalanced case, the distributional result in Equation (7) is approximate,[18] \(\tilde{Y}\) and \(SS_y\) are independently distributed, and \(SS_y\) and \(SS_e\) are independently distributed. However, \(\tilde{Y}\) and \(SS_y\) are not independent. As noted in Krishnamoorthy and Mathew,[8] and as will be seen in the sequel, this dependence does not pose any serious problems to obtain a satisfactory TLs for the unbalanced case.

### 3.1. One-sided TLs for \(N(\mu, \sigma^2 + \sigma^2_e)\)

#### 3.1.1. MOVER TLs

To construct a TL for the \(N(\mu, \sigma^2 + \sigma^2_e)\) distribution, let
\[
\sigma^2_1 = \sigma^2 + \tilde{n}\sigma^2_e \quad \text{and} \quad \sigma^2_2 = \sigma^2_e, \quad \text{so that} \quad \sigma^2 = \sigma^2_1 - \tilde{n}\sigma^2_2.
\]

In these notations, we see that \(Y_{ij} \sim N(\mu, \sigma^2 + \sigma^2_e) = N(\mu, a_1\sigma^2_1 + a_2\sigma^2_2)\) with \((a_1, a_2) = (1, 1 - \tilde{n})\), and \(\hat{\mu} = \tilde{Y} \sim N(\mu, \sigma^2_1/k)\). To obtain a \(1 - \alpha\) upper CL for \(a_1\sigma^2_1 + a_2\sigma^2_2\), note that
\[
\hat{\sigma}^2_1 = \frac{SS_y}{k - 1} \sim \frac{\chi^2_{k-1}}{k-1} \quad \text{and} \quad \hat{\sigma}^2_2 = \frac{SS_e}{N - k} \sim \frac{\chi^2_{N-k}}{N - k}.
\]

Furthermore, \(\hat{\sigma}^2_1\) and \(\hat{\sigma}^2_2\) are independent with \(E(\hat{\sigma}^2_1) = \sigma^2_1\) and \(E(\hat{\sigma}^2_2) = \sigma^2_2\). Using these distributional results, we obtain a \(1 - \alpha\) upper CL for \(\mu\) as
\[
U_\mu = \tilde{Y} + T_{k-1,1-\alpha} \frac{\hat{\sigma}_1}{\sqrt{k}}.
\]

The above CL is exact when the data are balanced, and is an approximate when the data are unbalanced. The upper CL for \(a_1\sigma^2_1 + a_2\sigma^2_2\) can be obtained as
\[
U_\nu = (a_1\hat{\sigma}^2_1 + a_2\hat{\sigma}^2_2) + \left(a_1^2\hat{\sigma}^4_1 \left(\frac{k - 1}{\chi^2_{k-1,\alpha}} - 1\right) + a_2^2\hat{\sigma}^4_2 \left(\frac{N - k}{\chi^2_{N-k,\alpha}} - 1\right)\right)^{1/2}.
\]

\[\text{(10)}\]
Using Equations (9) and (10) in Equation (3), a \((p, 1 - \alpha)\) upper TL for \(N(\mu, \sigma_t^2 + \sigma_e^2)\) can be obtained as

\[
\tilde{Y} + \left\{ z_p(a_1\hat{\sigma}_t^2 + a_2\hat{\sigma}_e^2)^{1/2} + \left( \frac{t_{k-1,1-\alpha}}{k} \right) \right\},
\]

where \(U_v\) is given in Equation (10), and \(a_1 = 1\) and \(a_2 = 1 - \bar{n}\). Furthermore, it follows from Equation (4) that the \((p, 1 - \alpha)\) MOVER lower TL for the \(N(\mu, \sigma_t^2 + \sigma_e^2)\) is given by

\[
\tilde{Y} - \left\{ z_p(a_1\hat{\sigma}_t^2 + a_2\hat{\sigma}_e^2)^{1/2} + \left( \frac{t_{k-1,1-\alpha}}{k} \right) \right\}.
\]

### 3.1.2. Tls based on noncentral t percentiles

The approximate upper TL given in Krishnamoorthy and Mathew [8] is expressed as

\[
\tilde{Y} + t_{k-1,1-\alpha}(\delta) \sqrt{\frac{SS_{\tilde{Y}}}{k(k-1)}}, \quad \text{with } \delta = z_p \left( k + \frac{k(k-1)(1-\bar{n})SS_e}{N-k} \right)^{1/2},
\]

where \(F_{m,n,q}\) denotes the \(q\)th quantile of an \(F\) distribution with dfs \(m\) and \(n\). We shall refer to the above TL as the NCT TL.

### 3.1.3. Vangel’s Tls

Vangel [6] proposed approximate Tls for the balanced case, and they can be described as follows. Define

\[
\tilde{Y}_i = \frac{1}{n} \sum_{j=1}^{n} Y_{ij}, \quad \tilde{Y} = \frac{1}{kn} \sum_{i=1}^{k} \sum_{j=1}^{n} Y_{ij}, \quad SS_\tau = \sum_{i=1}^{k} \sum_{j=1}^{n} (\tilde{Y}_i - \bar{Y})^2 \quad \text{and} \quad SS_e = \sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \tilde{Y}_i)^2.
\]

Further, define \(MS_\tau = SS_\tau / (k-1)\), \(MS_e = SS_e / (n-1)\), \(\hat{\sigma}_\tau^2 = (1/n)(MS_\tau - MS_e)\) and \(\hat{\sigma}_e^2 = MS_e\). Vangel’s lower TL is expressed as \(\tilde{Y} - F \sqrt{\hat{\sigma}_\tau^2 + \hat{\sigma}_e^2}\), and the corresponding upper TL is given by \(\tilde{Y} + F \sqrt{\hat{\sigma}_\tau^2 + \hat{\sigma}_e^2}\), where the \(F\) is the tolerance factor given by

\[
F_{V_e} = a_1 + a_2W + a_3W^2 + a_4W^3,
\]

and \(W = (1 + (n-1)/(MS_\tau / MS_e))^{-1/2}\). The coefficients \(a_1, a_2, a_3\) and \(a_4\) are to be determined numerically, and they are tabulated in Vangel’s paper for \((p, 1 - \alpha) = (0.90, 0.95)\) and \((0.99, 0.95)\), and some selected values of \((k, n)\).

### 3.2. One-sided Tls for \(N(\mu, \sigma_t^2)\)

#### 3.2.1. MOVER Tls

To construct a \((p, 1 - \alpha)\) upper TL for \(N(\mu, \sigma_t^2)\), we note from Equation (8) that \(\sigma_t^2 = a_1\sigma_1^2 + a_2\sigma_2^2\) with \(a_1 = 1\) and \(a_2 = -\bar{n}\). To find an upper CL for \(\mu + z_p\sigma_\tau = \mu + z_p\sqrt{a_1\sigma_1^2 + a_2\sigma_2^2}\), we can
use the upper CL for \( \mu \) as defined in Equation (9), and we need to find only the upper CL for \( a_1 \sigma^2_1 + a_2 \sigma^2_2 \) with \( a_1 = 1 \) and \( a_2 = -\hat{n} \). The \( 1 - \alpha \) upper CL for \( a_1 \sigma^2_1 + a_2 \sigma^2_2 \) is expressed as

\[
U_v = (a_1 \hat{\sigma}^2_1 + a_2 \hat{\sigma}^2_2) + \frac{a_1^2 \sigma_1^4}{\chi^2_{a-1,\alpha}} + a_2^2 \sigma_2^4 \left( \frac{N - a}{\chi^2_{N - a, 1 - \alpha}} - 1 \right)^2 \right)^{1/2}.
\]

In terms of \( U_\mu \) in Equation (9) and \( U_v \) in Equation (15), the \((p, 1 - \alpha)\) upper TL for \( N(\mu, \sigma^2) \) is given by

\[
\tilde{Y} + \left\{ z_p (a_1 \hat{\sigma}^2_1 + a_2 \hat{\sigma}^2_2)^{1/2} + \left( t_{k-1,1-a} \hat{\sigma}_1^2 + z^2_p \left( \frac{(a_1 \hat{\sigma}^2_1 + a_2 \hat{\sigma}^2_2)^{1/2} - \sqrt{U_v}}{\sqrt{2}} \right) \right) \right\}.
\]

where \( a_1 = 1 \) and \( a_2 = -\hat{n} \). The \((p, 1 - \alpha)\) lower TL is given by Equation (16) with the plus sign after \( \tilde{Y} \) replaced by the minus sign.

### 3.2.2. Noncentral t TLs

The \((p, 1 - \alpha)\) upper TL for a \( N(\mu, \sigma^2) \) distribution based on the noncentral \( t \) approximation [8] is given by

\[
\tilde{Y} + t_{k-1,1-a}(\delta) \sqrt{\frac{SS_y}{k(k-1)}},
\]

where

\[
\delta = z_p \left\{ k - \frac{\hat{n}k(k-1) SS_y}{N-k SS_y F_{k-1,N-k,\alpha}} \right\}^{1/2}.
\]

### 3.3. CLs for a survival probability

Consider the one-way random model in Equation (5). Let \( \theta = P(Y > T) \), where \( T \) is a specified value, and \( Y \sim N(\mu, \sigma^2 + \sigma^2) \). In industrial hygiene, \( T \) is the occupational exposure limit (OEL), and the probability is referred to as the exceedance probability. In the manufacturing industry, this probability is referred to as the unilateral conformance proportion, which is an important index for assessing the efficiency of a manufacturing process; see Lee and Liao.\[19\] A \( 1 - \alpha \) lower CL for \( \theta \) is the value of \( p \) for which the \((p, 1 - \alpha)\) lower TL is equal to \( T \). That is, the value of \( p \) that satisfies the equation

\[
\tilde{Y} - z_p \sqrt{a_1 \hat{\sigma}^2_1 + a_2 \hat{\sigma}^2_2} - \left( t_{k-1,1-a} \hat{\sigma}_1^2 + z^2_p (a_1 \hat{\sigma}^2_1 + a_2 \hat{\sigma}^2_2 - \sqrt{U_v})^2 \right)^{1/2} = T.
\]

It is easy to check that \( z_p \) satisfies the above equation and it is given by

\[
A = \frac{(b - \sqrt{b^2 - ac})}{a},
\]

where \( a = a_1 \hat{\sigma}^2_1 + a_2 \hat{\sigma}^2_2 - (\sqrt{a_1 \hat{\sigma}^2_1 + a_2 \hat{\sigma}^2_2 - \sqrt{U_v})^2, b = (a_1 \hat{\sigma}^2_1 + a_2 \hat{\sigma}^2_2)^{1/2}(\tilde{Y} - T) \) and \( c = (\tilde{Y} - T)^2 - t_{k-1,1-a} \hat{\sigma}_1^2 / k \), and \( U_v \) is given in Equation (10). The approximate lower CL for \( \theta \) is given by \( \Phi(A) \), where \( \Phi \) denotes the cumulative distribution function of the standard normal random variable.
There are practical situations where one is interested in finding an upper CL for $\theta$. For instance, in exposure data analysis, it is desired to find upper CL for the probability that the exposure level of a random worker exceeds the OEL. A $1 - \alpha$ upper CL for $\theta$ is obtained similarly by equating $(p, 1 - \alpha)$ upper TL to $T$, and solving the resulting equation for $p$. This yields upper CL for $\theta$ as $\Phi(A^*)$, with $A^* = (-b - \sqrt{b^2 - ac})/a$, where $a$, $b$, and $c$ are as defined in the preceding paragraph.

### 3.4. Coverage studies

To determine the performance of the new TLs described in the preceding sections, and to compare with the noncentral $t$ TLs, we estimated the coverage probabilities and expectations of both TLs by Monte Carlo method with 100,000 runs. Estimates are obtained for various values of $k$, $n$ and the intraclass correlation $\rho = \sigma_\tau^2 / (\sigma_\tau^2 + \sigma_\epsilon^2)$. In our simulation study, without loss of generality, we set $\mu = 0$ and $\sigma_\tau^2 + \sigma_\epsilon^2 = 1$, so that $\mu + z_p \sqrt{\sigma_\tau^2 + \sigma_\epsilon^2} = z_p$. The results in Table 1 correspond to $(.90, .95)$ upper TLs based on balanced data. Furthermore, we note that the sample size and parameter configurations are chosen as in Table 2 of Krishnamoorthy and Mathew.[8] From Table 1, we see that the MOVER TLs are conservative for small $k$, $n$ and $\rho < .5$. Even though the coverage probabilities of the MOVER TLs are larger than .95, their expected values are comparable, and shorter than the corresponding NCT and Vangel’s TLs for some cases. For example, for $(k, n, \rho) = (3, 4, .5)$. In situations where the coverage probabilities of all TLs

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Table 1. Monte carlo estimates of the coverage probabilities and expected values for the $(p, 1 - \alpha)$ NCT, MOVER and Vangel’s upper TLs for $N(\mu, \sigma_\tau^2 + \sigma_\epsilon^2)$ based on balanced data with $\mu = 0$, $\rho = \sigma_\tau^2 / (\sigma_\tau^2 + \sigma_\epsilon^2)$, $\sigma_\tau^2 + \sigma_\epsilon^2 = 1$, $(p, \gamma) = (0.90, 0.95)$; the expected values are given in parentheses.
Table 2. Monte Carlo estimates of coverage probabilities and expected value of the (.90, .95) upper TL for $N(\mu, \sigma^2 + \sigma_T^2)$ based on unbalanced data with $\mu = 0$, $\rho = \sigma_T^2/(\sigma^2 + \sigma_T^2)$, $\sigma^2 + \sigma_T^2 = 1$; the expected values are given in parentheses.

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Notes: a, $n = (3, 15, 30, 14, 2, 3, 13, 22, 8, 6, 9, 11)$; b, $n = (3, 4, 3, 2, 3, 3, 2, 2, 2, 2, 2, 2)$; c, $n = (13, 40, 7, 14, 22, 30, 3, 2, 12, 2, 21, 4)$; d, $n = (20, 30, 40, 30, 20, 10, 40, 20, 10, 5, 32, 40)$.

are close to the nominal level, the MOVER TLs are expected to be smaller than the other TLs; see for the cases $(k, n, \rho) = (3, 4, \geq .7)$ and $(3, 10^3, \geq .5)$. For moderate-to-large samples, the MOVER TLs are superior to the NCT TLs in terms of coverage probabilities and precision. We also observe that MOVER TL is quite comparable with the Vangel TL in most cases in terms of coverage probabilities and precision. However, the MOVER TLs are simple to compute and does not require any table values, whereas the Vangel TLs require table values (the coefficients $a_i$‘s in Equation (14)) and are available only for the balanced case.

The estimated coverage probabilities reported in Table 2 are for the unbalanced data. We once again see that the NCT TLs could be very liberal for smaller values of $\rho$ where as the MOVER TLs are conservative. Nevertheless, the expected values of the MOVER TLs are comparable or smaller than those of the NCT TLs. For example, when $(\rho, n_1, n_2, n_3) = (.2, 4, 6, 7)$, the coverage probability of the MOVER TL is .969 with the expected value 3.45, and those of the NCT TL are .942 and 3.45, respectively. For $\rho \geq .5$, which could be the case in many applications, both TLs are satisfactory. Even in these cases, the expected values of the MOVER TLs are slightly smaller than the corresponding NCT TLs.
Table 3. Monte Carlo estimates of the coverage probabilities and expected values for the (0.90, 0.95) NCT and MOVER upper TLs for N(μ, σ^2) based on balanced data with μ = 0, ρ = σ^2 / (σ^2 + σ^2), σ^2 + σ^2 = 1; the expected values are given in parentheses.

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<th>(k, n)</th>
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<th>MOVER</th>
<th>(k, n)</th>
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Estimates of coverage probabilities and expected values of upper TLs for N(μ, σ^2) are given in Table 3 (balanced case) and Table 4 (unbalanced case). These two table values indicate that both TLs are conservative for small values of ρ, and control the coverage probability satisfactorily for ρ ≥ .5. We see in general that the expected values of the MOVER TLs are slightly smaller than those of NCT TLs.

Overall, we see that the MOVER TLs are not only simple to compute, but also they are very satisfactory. In many cases, we noticed that the MOVER TLs are better than those based on other methods.

4. A two-way nested model

Consider the design of experiment where there are two factors A and B with the levels of B nested within the levels of A. Suppose there are a levels of A, and b levels of B are nested within each level of A, with n observations obtained on each combination of levels. Let Y_{ijl} denote the lth observation corresponding to the jth level of B nested within the ith level of A. Thus, we have

\[ Y_{ijl} = \mu + \tau_i + \beta_{j(i)} + e_{ijl}, \quad i = 1, \ldots, a; \ j = 1, \ldots, b; \ l = 1, \ldots, n, \quad (19) \]

where μ is an overall mean effect, τ_i is the main effect due to the ith level of A, \( \beta_{j(i)} \) is the effect due to the jth level of B nested within the ith level of A, and \( e_{ijl} \)'s are the error terms with
Table 4. Monte carlo estimates of the confidence level and expected value of the (90, 95) NCT and MOVER upper TLs for $N(\mu, \sigma^2)$ based on unbalanced data with $\mu = 0.0, \sigma^2 = 1.0, Q = \mu + \beta \sigma_1, \rho = \sigma^2 / (\sigma_1^2 + \sigma^2); \text{ the expected values are given in parentheses.}

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<td>.999</td>
<td>.987</td>
<td>.999</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.95)</td>
<td>(2.93)</td>
<td>(1.21)</td>
<td>(1.20)</td>
<td></td>
</tr>
</tbody>
</table>

$e_{ijl} \sim N(0, \sigma^2)$. If the levels of A and B are randomly selected, then we have a random effects model, and in addition to the above distributional assumptions, we also assume $\tau_i \sim N(0, \sigma^2_{\tau})$ and $\beta_{i(j)} \sim N(0, \sigma^2_{\beta})$, and $\tau_i, \beta_{i(j)}$ and $e_{ijl}$ are all independent. The required statistics to find the MOVER TLs are defined as follows:

$$
\bar{Y}_{ij} = \frac{1}{n} \sum_{l=1}^{n} Y_{ijl}, \quad \bar{Y}_{i.} = \frac{1}{bn} \sum_{j=1}^{b} \sum_{l=1}^{n} Y_{ijl}, \quad \bar{Y}_{..} = \frac{1}{abn} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{l=1}^{n} Y_{ijl},
$$

$$
SS_\tau = bn \sum_{i=1}^{a} (\bar{Y}_{i.} - \bar{Y}_{..})^2, \quad SS_\beta = n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{Y}_{ij} - \bar{Y}_{i.})^2, \quad \text{and } SS_\epsilon = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{l=1}^{n} (Y_{ijl} - \bar{Y}_{ijl})^2.
$$

Let $\sigma^2_1 = b n \sigma^2 + n \sigma^2_\epsilon + \sigma^2_\epsilon, \sigma^2_2 = n \sigma^2_\epsilon + \sigma^2_\epsilon$ and $\sigma^2_3 = \sigma^2_\epsilon$ so that $\sigma^2_\tau = (\sigma^2_1 - \sigma^2_2)/(bn), \sigma^2_\beta = (\sigma^2_2 - \sigma^2_3)/n$ and $\sigma^2_\epsilon = \sigma^2_3$. Under the model assumptions, we have

$$
\hat{\mu} = \bar{Y}_{..} \sim N \left( \mu, \frac{\sigma^2_1}{abn} \right), \quad \frac{SS_\tau}{\sigma^2_1} \sim \chi^2_{a-1}, \quad \frac{SS_\beta}{\sigma^2_2} \sim \chi^2_{ab-1}, \quad \text{and } \frac{SS_\epsilon}{\sigma^2_3} \sim \chi^2_{ab(n-1)},
$$

where all the random variables are mutually independent. It follows from the above distributional results that

$$
\hat{\sigma}^2_1 = \frac{SS_\tau}{a - 1}, \quad \hat{\sigma}^2_2 = \frac{SS_\beta}{a(b - 1)}, \quad \text{and } \hat{\sigma}^2_3 = \frac{SS_\epsilon}{ab(n - 1)}
$$

are independent unbiased estimators of $\sigma^2_1, \sigma^2_2$ and $\sigma^2_3$, respectively.
4.1. **TLs for** \( N(\mu, \sigma^2 + \sigma^2_\beta + \sigma^2_e) \)

### 4.1.1. **MOVER TLs**

We shall find \((p, 1 - \alpha)\) one-sided upper TL for the distribution \( N(\mu, \sigma^2 + \sigma^2_\beta + \sigma^2_e) \), equivalently, a \( 1 - \alpha \) upper CL for \( \mu + z_p \sqrt{\sum a_i \sigma_i^2} \) where \( a_1 = 1/(bn) \), \( a_2 = (1 - 1/b)/n \) and \( a_3 = (1 - 1/n) \). The upper CL for \( \mu \) is

\[
U_\mu = \bar{Y}_- + t_{a-1,\beta} \frac{\hat{\sigma}_1}{\sqrt{abn}}, \tag{22}
\]

and the MOVER upper CL for \( \sum a_i \sigma_i^2 \) is given by

\[
U_v = \sum a_i \hat{\sigma}_i^2 + \sqrt{\sum a_i \hat{\sigma}_i^4 \left( \frac{m_i}{\chi^2_{m_i,\alpha}} - 1 \right)^2}, \tag{23}
\]

where \( m_1 = a - 1, m_2 = a(b - 1) \) and \( m_3 = ab(n - 1) \). Finally, the \((p, 1 - \alpha)\) MOVER upper TL for the \( N(\mu, \sum a_i \sigma_i^2) \) distribution is expressed as

\[
\bar{Y}_- + z_p \left( \sum a_i \hat{\sigma}_i^2 \right)^{1/2} + \left[ t_{a-1,1-\alpha} \frac{\hat{\sigma}_1}{\sqrt{abn}} + z_p \sqrt{\sum a_i \hat{\sigma}_i^2 - \sqrt{U_v}} \right]^{1/2}. \tag{24}
\]

A \((p, 1 - \alpha)\) lower TL is obtained from Equation (24) with the plus sign after \( \bar{Y}_- \) replaced by the minus sign.

### 4.1.2. **Generalized TLs**

The \((p, 1 - \alpha)\) upper TL based on the generalized variable approach [16] is expressed as follows. Let \((\bar{Y}_-, ss_T, ss_\beta, ss_e)\) be an observed value of \((\bar{Y}_-, SS_T, SS_\beta, SS_e)\). In terms of these notations, the generalized pivotal quantity is given by

\[
\bar{Y}_- + \frac{Z}{\sqrt{\chi^2_{a-1}}} \sqrt{\frac{ss_T}{abn}} + \frac{z_p}{\sqrt{bn}} \left[ \frac{ss_T}{\chi^2_{a-1}} + (b - 1) \frac{ss_\beta}{\chi^2_{a(b-1)}} + b(n - 1) \frac{ss_e}{\chi^2_{ab(n-1)}} \right]^{1/2}, \tag{25}
\]

where \( Z \sim N(0, 1) \), and all the random variables in Equation (25) are mutually independent. The 100\((1 - \alpha)\) percentile is the desired upper TL for the \( N(\mu, \sigma^2_T + \sigma^2_\beta + \sigma^2_e) \) distribution. For a given \((\bar{Y}_-, ss_T, ss_\beta, ss_e)\), the percentiles of Equation (25) can be estimated using Monte Carlo simulation.

### 4.2. **TLs for** \( N(\mu, \sigma^2_T + \sigma^2_\beta) \)

#### 4.2.1. **MOVER TLs**

An upper TL for the distribution \( N(\mu, \sigma^2_T + \sigma^2_\beta) \) can be readily obtained from Equation (24) as follows. Note that \( \sigma^2_\beta + \sigma^2_T = a_1 \sigma^2_1 + a_2 \sigma^2_2 + a_3 \sigma^2_3 \), where \( a_1 = (bn)^{-1} \), \( a_2 = (b - 1)/(nb) \), and
\( a_3 = -1/n \). The MOVER upper CL for \( \sum_{i=1}^{3} a_i \sigma_i^2 \) is given by

\[
U_v = \sum_{i=1}^{3} a_i \hat{\sigma}_i^2 + \left[ a_1^2 \hat{\sigma}_1^4 \left( \frac{m_1}{\chi_{m_1 \alpha}^2} - 1 \right) + a_2^2 \hat{\sigma}_2^4 \left( \frac{m_2}{\chi_{m_2 \alpha}^2} - 1 \right) + a_3^2 \hat{\sigma}_3^4 \left( \frac{m_3}{\chi_{m_3;1-\alpha}^2} - 1 \right) \right]^{1/2}, \tag{26}
\]

where \( m_i \)'s are as defined in Equation (24). The MOVER upper TL for the distribution \( N(\mu, \sigma_T^2 + \sigma_\beta^2) \) is given by Equation (24) with \( U_v \) defined in Equation (26).

### 4.2.2. Generalized TLs

The generalized upper TL due to Fonseca et al. [16] is given as follows. Let \( (\bar{Y}_-, SS_\tau, SS_\beta, SS_e) \) be an observed value of \( (\bar{Y}_-, SS_\tau, SS_\beta, SS_e) \). In terms of these notations, the generalized pivotal quantity is given by

\[
\bar{y}_- + \frac{Z}{\sqrt{\chi_{a-1}^2}} \left[ \frac{SS_\tau}{bn} + \frac{SS_\beta}{\chi_{ab(b-1)}^2} - b \frac{SS_e}{\chi_{ab(b-1)}^2} \right]^{1/2}, \tag{27}
\]

where \( Z \sim N(0, 1) \), and all the random variables in Equation (27) are mutually independent. The 100(1 - \( \alpha \)) percentile is the desired upper TL for the \( N(\mu, \sigma_T^2 + \sigma_\beta^2) \) distribution. For a given \( (\bar{y}_-, SS_\tau, SS_\beta, SS_e) \), the percentiles of Equation (27) can be estimated using Monte Carlo simulation.

### 4.3. Coverage studies

In order to judge the performance of the MOVER TLs for the two-way nested model, and to compare them with the generalized TLs proposed in Fonseca et al.,[16] we estimated the coverage probabilities of the (.90, .95) TLs using Monte Carlo simulation consisting of 100,000 runs. For simulation studies, we used the same sample size and parameter configurations given in Fonseca et al.[16] Specifically, the simulations were carried out fixing \( n = 20 \), and the quantities \( a \) and \( b \) were chosen to be 10 or 20. In Table 5, we reported the estimated coverage probabilities of

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>.1</th>
<th>.3</th>
<th>.5</th>
<th>.7</th>
<th>.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = 5, b = 5 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GV</td>
<td>.974</td>
<td>.970</td>
<td>.965</td>
<td>.959</td>
<td>.952</td>
</tr>
<tr>
<td>MOVER</td>
<td>.967</td>
<td>.960</td>
<td>.959</td>
<td>.958</td>
<td>.953</td>
</tr>
<tr>
<td>( a = 5, b = 20 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GV</td>
<td>.969</td>
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<td>.961</td>
<td>.957</td>
<td>.952</td>
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<tr>
<td>MOVER</td>
<td>.958</td>
<td>.957</td>
<td>.957</td>
<td>.957</td>
<td>.957</td>
</tr>
<tr>
<td>( a = 20, b = 5 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GV</td>
<td>.972</td>
<td>.968</td>
<td>.967</td>
<td>.965</td>
<td>.958</td>
</tr>
<tr>
<td>MOVER</td>
<td>.955</td>
<td>.953</td>
<td>.952</td>
<td>.952</td>
<td>.952</td>
</tr>
<tr>
<td>( a = 20, b = 20 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GV</td>
<td>.963</td>
<td>.961</td>
<td>.959</td>
<td>.959</td>
<td>.957</td>
</tr>
<tr>
<td>MOVER</td>
<td>.951</td>
<td>.951</td>
<td>.952</td>
<td>.952</td>
<td>.952</td>
</tr>
</tbody>
</table>

Notes: \( \sigma_T^2 = \sigma_e^2 = 1; \rho = \sigma_T^2/(\sigma_T^2 + \sigma_\beta^2 + \sigma_e^2); n = 20. \)
Table 6. Coverage probabilities of (.90, .95) upper TLs for \(N(\mu, \sigma^2_r + \sigma^2_e)\) in a two-way nested model.

<table>
<thead>
<tr>
<th>(\rho)</th>
<th>.1</th>
<th>.3</th>
<th>.5</th>
<th>.7</th>
<th>.9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a = 5, b = 5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GV</td>
<td>.977</td>
<td>.972</td>
<td>.967</td>
<td>.959</td>
<td>.952</td>
</tr>
<tr>
<td>MOV\R</td>
<td>.963</td>
<td>.954</td>
<td>.949</td>
<td>.946</td>
<td>.945</td>
</tr>
<tr>
<td>(a = 5, b = 20)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>GV</td>
<td>.972</td>
<td>.966</td>
<td>.960</td>
<td>.956</td>
<td>.951</td>
</tr>
<tr>
<td>MOV\R</td>
<td>.954</td>
<td>.948</td>
<td>.945</td>
<td>.945</td>
<td>.945</td>
</tr>
<tr>
<td>(a = 20, b = 5)</td>
<td></td>
<td></td>
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<tr>
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<td>.957</td>
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<tr>
<td>(a = 20, b = 20)</td>
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<td></td>
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<td>.960</td>
<td>.958</td>
<td>.957</td>
</tr>
<tr>
<td>MOV\R</td>
<td>.947</td>
<td>.945</td>
<td>.944</td>
<td>.945</td>
<td>.943</td>
</tr>
</tbody>
</table>

Notes: \(\sigma^2_r = \sigma^2_e = 1; \rho = \sigma^2_r/(\sigma^2_r + \sigma^2_e + \sigma^2_e); n = 20\).

MOVER TLs for \(N(\mu, \sigma^2_r + \sigma^2_e)\) along with those for the generalized TLs (reported in [16, Table 2]) in Table 5. The estimated coverage probabilities clearly indicated that the MOVER TLs are less conservative than the generalized TLs. For the case of \(a = b = n = 20\), the coverage probabilities of the MOVER TLs are very close to the nominal level .95.

The estimated coverage of (.90, .95) TLs for \(N(\mu, \sigma^2_r + \sigma^2_e)\) are given in Table 6. The coverage probabilities of the generalized TLs are taken from Table 2 of Fonseca et al.[16] We once again note that the MOVER TLs are less conservative than the generalized TLs, and in some cases the coverage probabilities are slightly less than .95, but not less than .94. Overall, we see that the MOVER TLs are less conservative than the generalized TLs, and they do not require simulation. So the MOVER TLs are preferable to the generalized TLs in terms of accuracy and simplicity.

5. Examples

Example 1  This example is taken from Vangel,[6] and is concerned with tensile strength measurements made on \(k = 5\) batches of composite materials. Each batch consists of \(n = 5\) specimens. The data are given in Vangel [6, Table 4] and they are reproduced here in Table 7. The summary statistics to compute TLs are \(\bar{Y}_1 = 375.6, \bar{Y}_2 = 370.4, \bar{Y}_3 = 401.6, \bar{Y}_4 = 394.0, \bar{Y}_5 = 400.2, \bar{Y} = 388.36, SS_{\bar{Y}} = 832.67, \) and \(SS_e = 1578.4\). Other quantities to compute the (.90, .95) MOVER lower TL are

\[
a_1 = 1, \quad a_2 = .8, \quad \hat{\sigma}^2_1 = 208.17, \quad \hat{\sigma}^2_2 = 78.92, \quad \hat{\sigma}^2_e = 4.5448 \quad \text{and} \quad U_v = 1236.20,\]

where \(U_v\) is computed using Equation (9). Using these values along with \(z_9 = 1.282\) in Equation (12), we found the desired lower TL as 339.63. This limit along with the ones [8] based on the generalized variable approach, noncentral \(t\) approximation, and Vangel’s [6] approach are given in Table 7.

Note that the MOVER TL is also the largest among the four. Since we are computing a lower TL, the larger the TL, the better.

Suppose it is desired to assess the percentage of tensile strength measurements that exceed 350. The required quantities (see Equation (18)) are \(a = -77.9444, b = 631.849, c = 1282.27, \) and

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Suppose it is desired to assess the percentage of tensile strength measurements that exceed 350. The required quantities (see Equation (18)) are \(a = -77.9444, b = 631.849, c = 1282.27, \) and
Table 7. The tensile strength data and lower TLs.

<table>
<thead>
<tr>
<th>Batch</th>
<th>Observations</th>
<th>Methods</th>
<th>(.90,.95) lower TLs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>379</td>
<td>357</td>
<td>390</td>
</tr>
<tr>
<td>2</td>
<td>363</td>
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<td>402</td>
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</tr>
<tr>
<td>4</td>
<td>402</td>
<td>387</td>
<td>392</td>
</tr>
<tr>
<td>5</td>
<td>415</td>
<td>405</td>
<td>396</td>
</tr>
</tbody>
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Table 8. The moisture content data.

<table>
<thead>
<tr>
<th>Storage condition</th>
<th>Moisture content (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>5.4, 7.4, 7.1</td>
</tr>
<tr>
<td>3</td>
<td>8.1, 6.4</td>
</tr>
<tr>
<td>4</td>
<td>7.9, 9.5, 10.0</td>
</tr>
<tr>
<td>5</td>
<td>7.1</td>
</tr>
</tbody>
</table>

\[ A = .95810. \] The 95% lower confidence for the exceedance probability is \( \Phi(.9581) = .831. \) This means that at least 83% of tensile strengths exceed 350 with confidence 95%.

**Example 2** This example is taken from Ostle and Mensing [20, p. 296] and is on a study of the effect of storage conditions on the moisture content of white pine lumber. Five different storage conditions (\( k = 5 \)) were studied with a varying number of sample boards stored under each condition. The example is also considered by Bhaumik and Kulkarni,[5] and the data are reproduced here in Table 8. These authors assumed the one-way random model for the purpose of estimating a (0.90, 0.95) upper TL for moisture content. We see from Table 8 that \( n_1 = 5, n_2 = 3, n_3 = 2, n_4 = 3 \) and \( n_5 = 1. \) The summary statistics are

\[
\bar{Y} = 7.62, \quad SS_{\bar{Y}} = 3.80, \quad SS_e = 7.17, \quad \bar{n} = .4733, \quad \hat{\sigma}_1^2 = .9500, \quad \text{and } \hat{\sigma}_2^2 = .7967.
\]

To compute the MOVER upper TL, we found the .95 upper CL for \( \mu \) as 8.5493 and the .95 upper CL for \( a_1\sigma_1^2 + a_2\sigma_2^2 \) as 5.8242, where \( a_1 = 1 \) and \( a_2 = 1 - \bar{n}. \) Using these values along with \( z_{.9} = 1.282, \) we calculated the MOVER upper TL in Equation (16) as 10.96. The NCT upper TL and the one based on the generalized variable approach are reported in Krishnamoorthy and Mathew [8] as 11.04 and 11.12, respectively.

We also evaluated (0.90, 0.95) upper TL for the distribution of the ‘true’ moisture content, that is, the distribution of \( \mu + \tau_i, \) the MOVER upper TL is computed as 10.69. Krishnamoorthy and Mathew’s [8] calculation showed that the NCT TL and the one based on the generalized variable approach are very close, given by 10.94. We once again see that the MOVER TL is the smallest among these three TLs.

**Example 3** To show the applications of TLs in a two-way nested random model, we shall use the example of a breeding experiment given in Fonseca et al.[16] In this experiment, the breeding values of five sires in rising pigs were evaluated. Each sire was mated to two randomly selected dams, and the average weight gain of two pigs from each litter was recorded. Note that \( a = 5, b = 2 \) and \( n = 5. \) Other summary statistics are

\[
\bar{y}_{1..} = 2.67, \quad \bar{y}_{2..} = 2.53, \quad \bar{y}_{3..} = 2.63, \quad \bar{y}_{4..} = 2.47, \quad \bar{y}_{5..} = 2.57, \quad \text{ss}_\beta = .56, \quad \text{ss}_e = .39, \quad \text{ss}_\tau = .05.
\]
Assuming that both factors are random, Fonseca et al. obtained the (.90, .95) generalized upper TL for $N(\mu, \sigma^2 + \tau + \beta + e)$ as 3.08. To find the (.90, .95) MOVER upper TL, we calculated $U_v$ in Equation (23) as .1516, and $\bar{y} + z_p (\sum_{i=1}^{3} a_i \hat{\sigma}^2_i)^{1/2}$ as 2.8641. Using these numbers, and other statistics in Equation (24), we obtained the upper TL as 3.0797, which is the same as generalized upper TL 3.08.

Fonseca et al. [16] have calculated the (.90, .95) generalized upper TL for $N(\mu, \sigma^2 + \beta)$ as 2.99. To find the (.90, .95) MOVER upper TL, we calculated $U_v$ in Equation (26) as .1074 and $\bar{y} + z_p (\sum_{i=1}^{3} a_i \hat{\sigma}^2_i)^{1/2}$ as 2.7122. Finally, substituting these numbers in Equation (24), we obtained the upper TL as 2.99, which is the same as the generalized upper TL.

6. Concluding remarks

In this article, we have developed simple closed-form one-sided TLs in one-way random models with balanced or unbalanced data. Although the proposed MOVER method is a large sample method, our simulation studies clearly indicate that the method works satisfactorily even for small samples. We also notice that the TLs based on the MOVER method are not only simple to compute, but also comparable with those based on other methods and produce better TLs in some cases, especially when sample sizes are moderate to large. Overall, we observe in our simulation study that the MOVER TLs are seldom liberal, and so they can be safely used in applications.

It is evident that the performance of the MOVER TLs depends on the individual CIs for the overall mean effect $\mu$ and the variance components. Regarding the CI for $\mu$, we have chosen the usual one; however, there are choices for a linear combination of variance components. For example, one could use the generalized CIs proposed in Liao et al. [9] or the ones based on the Satterthwaite, [21] approximation to find a CI for $\sum a_i \sigma^2_i$. However, as noted in Graybill and Wang, [17] the MOVER CIs are better than the ones based the Satterthwaite approximation, and the MOVER method is applicable even when some $a_i$’s are negative. Our preliminary coverage studies of TLs (based on the Satterthwaite CIs of $\sum a_i \sigma^2_i$) indicated that such TLs are liberal for $\rho \geq .2$. The generalized CIs proposed in Liao et al. [9] are numerically involved, and require simulation to obtain them. Furthermore, as noted in the illustrative examples in Krishnamoorthy and Mathew [11, Chapters 4 and 5] and Krishnamoorthy and Lian, [22] coverage properties of the MOVER CIs and those of the generalized CIs for $\sum a_i \sigma^2_i$ should be similar. As the MOVER CIs for $\sum a_i \sigma^2_i$ are easy to compute, we recommend the TLs based on them.

References


