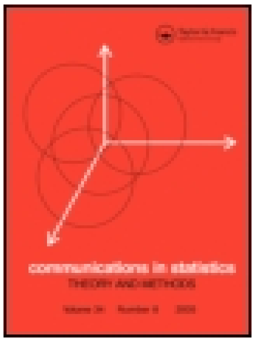

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Confidence intervals for a two-parameter exponential distribution: One- and two-sample problems

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ABSTRACT
The problems of interval estimating the mean, quantiles, and survival probability in a two-parameter exponential distribution are addressed. Distribution function of a pivotal quantity whose percentiles can be used to construct confidence limits for the mean and quantiles is derived. A simple approximate method of finding confidence intervals for the difference between two means and for the difference between two location parameters is also proposed. Monte Carlo evaluation studies indicate that the approximate confidence intervals are accurate even for small samples. The methods are illustrated using two examples.

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Coverage probability; Generalized pivotal quantity; MNA approximation; Tolerance limits.

1. Introduction

The two-parameter exponential distribution is a commonly used model in lifetime data analysis and reliability studies. Inferential methods for the exponential distribution, and applications in the context of life-testing and reliability, have been addressed in several articles and books; for example, see Bain and Engelhardt (1991), Balakrishnan and Basu (1995), Meeker and Escobar (1998), Lawless (2003), and the references therein. As the two-parameter exponential distribution is a member of the location–scale family, pivotal quantities based on the maximum likelihood estimates (MLEs) are easy to find, and pivotal-based approach to find exact test or confidence intervals (CIs) for the parameters and for the quantiles are readily available; for example see Lawless (2003, Chapter 4).

To review the pivotal-based approach, let $X_1, \ldots, X_n$ be a sample from a two-parameter exponential distribution with the probability density function

$$f(x|a, b) = \frac{1}{b}\exp(-(x-a)/b), \quad x > a, \ b > 0. \tag{1}$$

The MLEs of a and b are given by

$$\widehat{a} = X_{(1)} \ \text{and} \ \widehat{b} = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)}) = \bar{X} - X_{(1)}, \tag{2}$$

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where $X_{(1)}$ is the smallest of the X_i's. It is well known that the MLEs $\widehat{a}$ and $\widehat{b}$ are independent with

$$\frac{2n(\widehat{a}-a)}{b} \sim \chi^2_2 \text{ and } \frac{2n\widehat{b}}{b} \sim \chi^2_{2n-2}. \quad (3)$$

Closed-form CIs for a and b can be easily obtained on the basis of the above distributional results.

A pivotal quantity for estimating the mean or a quantile can be obtained as follows. Since the mean of the two-parameter exponential distribution is $\mu = a + b$, and the pth quantile is $a - \ln(1-p)b$, $0 < p < 1$, let us develop a pivotal quantity for a general case $a + cb$, where c is a known positive constant. Using the distributional results in (3), we find

$$\frac{a+cb-\widehat{a}}{\widehat{b}} = \frac{c-(\widehat{a}-a)/b}{\widehat{b}/b} \sim \frac{2nc-\chi^2_2}{\chi^2_{2n-2}} = f_{n,c}, \quad \text{say}, \quad (4)$$

where the Chi-square random variables are independent. For $0 < \alpha < 0.5$, let $f_{n,c;\alpha}$ denote the α quantile of $f_{n,c}$. Then

$$\left(\widehat{a} + f_{n,c;\alpha}\widehat{b},\ \widehat{a} + f_{n,c;1-\alpha}\widehat{b}\right) \quad (5)$$

is an exact $1 - 2\alpha$ CI for $a + cb$.

Guenther, Patil, and Uppuluri (1976) have evaluated the cumulative distribution function (cdf) of $f_{n,c}$ for calculating one-sided tolerance limits. Specifically, these authors have evaluated the cdf of $f_{n,c}$ when $c = -\log(p)$ and $c = -\ln(1-p)$, for calculating upper tolerance limits and lower tolerance limits, respectively. However, we found that the calculation of percentiles using their expression of the cdf poses some problems. Specifically, to find the percentiles using an iterative scheme, the initial value should be very close to the true percentile; otherwise, the iterative scheme could produce inaccurate percentiles. Since the calculation of exact tolerance limits by Guenther, Patil, and Uppuluri (1976) is numerically involved, Engehardt and Bain (1978) have proposed a large sample approximation for the lower and upper tolerance factors. However, our simulation studies (not reported here) indicated that such approximations are not satisfactory even for large samples of sizes around 50 or more. For estimating a survival probability, Roy and Mathew (2005) have developed a pivotal quantity using the *generalized variable approach* and proposed a simulation approach to obtain confidence limits. In general, a confidence limit for a survival probability can be deduced from appropriate one-sided tolerance limits. In fact, for the present setup, we can obtain an exact solution for the problem of estimating a survival probability from the derivation of the distribution function of $f_{n,c}$. Toward this, we give an alternate expression for the cdf of $f_{n,c}$ which is easy to compute, and it can be used to find the percentiles for calculating CIs for $a + cb$ for any given $c > 0$. In particular, the cdf can be used to find CIs for the mean and confidence bounds for quantiles.

Regarding two-sample problems, a test for the ratio of two scale parameters can be readily obtained from the distributional results in (3). Kumar and Patel (1971) have proposed an exact test for equality of two threshold (location) parameters assuming that the scale parameters are equal. Kharrati-Kopaei (2015), Kharrati-Kopaei, Malekzadeh, and Sadooghi-Alvandi (2013), and Maurya, Goyal, and Gill (2011) have developed simultaneous CIs for successive differences of location parameters, and a CI for the difference between two location parameters can be obtained as a special case. To the best of our knowledge, no interval estimation method or test was proposed for the difference between two means when the scale parameters are unknown and arbitrary.

The rest of the article is organized as follows. In the following section, we describe the recent modified normal-based approximation (MNA) by Krishnamoorthy (2016) to find the percentiles of a linear combination of independent random variables. In Section 3, we provide an exact expression for finding the cdf of $f_{n,c}$ in (4). In Section 4, we present CIs for the mean, one-sided confidence limits for quantiles and confidence limits for a survival probability. The necessary table values for constructing CIs for the mean and quantiles are also provided. In Section 5, we develop CIs for the difference between two location parameters when the scale parameters are unknown and arbitrary, and for the difference between two means using the generalized variable approach (Weerahandi 1993), and the MNA. The proposed CIs are evaluated in terms of coverage probabilities. The methods are illustrated using two examples in Section 6, and some concluding remarks are given in Section 7.

2. Modified normal-based approximations

Let $X_1, \ldots, X_k$ be independent continuous random variables not necessarily from the same family of distributions. Let $X_{i;\alpha}$ denote the α quantile of X_i, $i = 1, \ldots, k$. Let $Q = \sum_{i=1}^{k} w_i X_i$, where w_i's are known constants. Then, an approximation to the α quantile is given by

$$Q_\alpha \simeq \begin{cases} \sum_{i=1}^{k} w_i E(X_i) - \left[\sum_{i=1}^{k} w_i^2 [E(X_i) - X_i^*]^2\right]^{\frac{1}{2}}, & \text{for } 0 < \alpha \leq .5, \\ \sum_{i=1}^{k} w_i E(X_i) + \left[\sum_{i=1}^{k} w_i^2 [E(X_i) - X_i^*]^2\right]^{\frac{1}{2}}, & \text{for } 0.5 < \alpha < 1, \end{cases} \tag{6}$$

where $X_i^* = X_{i;\alpha}$ if $w_i > 0$ and is $X_{i;1-\alpha}$ if $w_i < 0$. Furthermore,

$$P(Q_{\alpha/2} \leq Q \leq Q_{1-\alpha/2}) \simeq 1 - \alpha, \quad \text{for } 0 < \alpha < 1.$$

Krishnamoorthy (2016) has obtained the above approximation, and it can be readily verified that the above approximate percentile in (6) is exact for normally distributed independent random variables. His numerical studies indicated that these MNAs are very satisfactory for many commonly used distributions such as the beta, Student's t, and the non central F.

An approximation to the percentiles of the ratio of two independent random variables can also be obtained from (6). Let X and Y be independent continuous random variables with means μ_x and μ_y, respectively, and assume that Y is a positive random variable. Let X_α (Y_α) denote the 100α percentile of X (Y), and let R_α denote the 100α percentile of X/Y. By equating the approximate 100α percentile of $X - R_\alpha Y$ to zero, and solving the equation for R_α, we find

$$R_\alpha \simeq \begin{cases} \dfrac{r - \left\{r^2 - \left[1 - \left(1 - \frac{Y_{1-\alpha}}{\mu_y}\right)^2\right]\left[r^2 - \left(r - \frac{X_\alpha}{\mu_y}\right)^2\right]\right\}^{\frac{1}{2}}}{\left[1 - \left(1 - \frac{Y_{1-\alpha}}{\mu_y}\right)^2\right]}, & 0 < \alpha \leq .5, \\ \dfrac{r + \left\{r^2 - \left[1 - \left(1 - \frac{Y_{1-\alpha}}{\mu_y}\right)^2\right]\left[r^2 - \left(r - \frac{X_{1-\alpha}}{\mu_y}\right)^2\right]\right\}^{\frac{1}{2}}}{\left[1 - \left(1 - \frac{Y_{1-\alpha}}{\mu_y}\right)^2\right]}, & .5 < \alpha < 1, \end{cases} \tag{7}$$

where $r = \mu_x/\mu_y$. For more details, see Krishnamoorthy (2016).

3. Evaluation of the cdf of $f_{n,c}$

As noted earlier, the percentiles of $f_{n,c} = (2nc - \chi^2)/\chi^2_{2n-2}$ (see expression (4)) with appropriate choices of c can be used to construct confidence bounds for the mean or quantiles.

To find the percentiles of $f_{n,c}$, let us first derive the cdf of $f_{n,c}$. As shown in the appendix,

$$P(f_{n,c} \le t) = \begin{cases} \frac{e^{-nc}}{(1-t)^{n-1}}, & t \le 0, \\ 1 - F_{2n}(2nc), & t = 1, \\ 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + \frac{e^{-nc}}{2^{n-1}\Gamma(n-1)} \int_0^{2nc/t} y^{n-2} e^{-0.5(1-t)y} dy, & 0 < t < 1 \text{ or } t > 1. \end{cases} \tag{8}$$

where $F_m(x)$ denotes the cdf of the χ^2_m random variable. Even though the above integral is finite and the integrand is continuous, standard integral routines (IMSL Gaussian quadrature, R function integrate()) produce erroneous results or "NaN" for some values of (t, n, c). For example, the percentiles of $f_{n,1}$ are needed to find CIs for the mean $a + b$, and the R function integrate() returned "NaN" when $(t, n) = (0.9999, 100)$, $(0.9999999, 50)$, and $(0.999999, 60)$. Furthermore, we require the percentiles of $f_{n,c}$ at $c = -\ln(1 - p)$, where $0 < p < 1$, to find one-sided tolerance limits. The R function integrate() returns "error in integrate()" for $(t, n, p) = (6, 90, 0.99)$, (5,100,0.99),(5,110,0.99), and (2,120,0.90). As the expression of the cdf in (8) poses computational problems for some values of (t, n, c), we shall provide an alternative expression which does not involve integral and can be computed for all values of (t, n, c).

Let $F_m(x)$ denote the cdf of the Chi-square distribution with degrees of freedom (dfs) $= m$. Then, we can express the cdf of $f_{n,c}$ as

$$P(f_{n,c} \le t) = \begin{cases} \frac{e^{-nc}}{(1-t)^{n-1}}, & t \le 0, \\ 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + e^{-\frac{nc}{t}} \sum_{m=0}^{N} \frac{(1-t)^m}{(m+n-1)!} \left(\frac{nc}{t}\right)^{m+n-1} + \text{error}, & \frac{1}{2} \le t \le \frac{3}{2}, \\ 1 - F_{2n-2}\left(\frac{2nc}{t}\right) & \\ + \frac{e^{-nc}}{(1-t)^{n-1}} \left(1 - e^{-\frac{nc(1-t)}{t}} \sum_{j=0}^{n-2} \frac{1}{j!} \left(\frac{nc(1-t)}{t}\right)^j\right), & 0 < t < \frac{1}{2} \text{ or } t > \frac{3}{2}, \end{cases} \tag{9}$$

where N is the smallest positive integer greater than or equal to $\max\{1, n(2e^2c - 1), \ln(\frac{1}{\sqrt{2\pi}\epsilon}) - n\}$ such that the "error" in (9) is less than the specified error tolerance ϵ. For derivation of the above cdf and details on computation of percentiles of $f_{n,c}$, see appendix.

4. One-sample problems

4.1. CIs for the mean

Let $f_{n,c;\alpha}$ denote the 100α percentile of $f_{n,c}$. Noting that the mean of the exponential distribution is $a + b$, an exact $1 - \alpha$ CI for the mean is (5) with $c = 1$. Specifically,

$$(\widehat{a} + f_{n,1;\alpha/2}\widehat{b},\ \ \widehat{a} + f_{n,1;1-\alpha/2}\widehat{b}) \tag{10}$$

is a $1 - \alpha$ CI for the mean. The percentiles $f_{n,1;\alpha}$ can be obtained numerically using the following approximate percentiles as starting values.

To find the approximate percentiles of $f_{n,1}$ using (7), let $X = 2n - \chi^2_2$ and $Y = \chi^2_{2n-2}$. Then, $\mu_x = \mu_y = 2n - 2$ gives $r = \mu_x/\mu_y = 1$, $X_\alpha = 2n - \chi^2_{2;1-\alpha}$, and $Y_{1-\alpha} = \chi^2_{2n-2;1-\alpha}$. Expression (7) with these quantities is an approximation to $f_{n,1;\alpha}$, and let us denote this approximate value by $f^*_{n,1;\alpha}$. These approximate percentiles can be used as starting values to find the exact percentiles of $f_{n,1}$ using a root bracketing method.

We have evaluated the approximate and exact percentiles of $f_{n,1}$ to calculate 90%, 95%, and 99% CIs for the mean. These percentiles are given in Table 1 for sample sizes ranging

Table 1. Exact and (approximate) percentiles of $f_{n,1}$ for constructing confidence intervals for the mean.

n	0.5%	2.5%	5%	95%	97.5%	99.5%
3	− 2.1466 (——)	− 0.4112 (——)	0.0021 (0.0021)	6.1249 (6.1749)	9.0370 (——)	21.068 (——)
4	− 0.5415 (——)	0.0985 (0.0940)	0.2847 (0.2711)	3.8942 (3.9479)	5.1717 (5.3064)	9.5308 (——)
5	− 0.0774 (−0.0847)	0.2795 (0.2599)	0.3984 (0.3844)	3.0613 (3.1033)	3.8533 (3.9546)	6.2830 (6.5729)
6	0.1309 (0.1209)	0.3717 (0.3494)	0.4643 (0.4520)	2.6286 (2.6614)	3.2008(3.2781)	4.8448 (5.0650)
7	0.2469 (0.2225)	0.4297 (0.4086)	0.5092 (0.4987)	2.3628 (2.3890)	2.8122 (2.8732)	4.0466 (4.2191)
8	0.3203 (0.2893)	0.4710 (0.4518)	0.5427 (0.5336)	2.1823 (2.2037)	2.5539 (2.6034)	3.5417 (3.6807)
9	0.3710 (0.3383)	0.5027 (0.4854)	0.5692 (0.5611)	2.0510 (2.0690)	2.3693 (2.4104)	3.1939 (3.3087)
10	0.4088 (0.3766)	0.5281 (0.5125)	0.5908 (0.5836)	1.9510 (1.9663)	2.2304(2.2652)	2.9395 (3.0364)
11	0.4384 (0.4076)	0.5493 (0.5352)	0.6089 (0.6025)	1.8720 (1.8852)	2.1218 (2.1517)	2.7453 (2.8283)
12	0.4627 (0.4337)	0.5673 (0.5545)	0.6245 (0.6187)	1.8079 (1.8194)	2.0344 (2.0604)	2.5918 (2.6639)
13	0.4831 (0.4559)	0.5829 (0.5712)	0.6381 (0.6328)	1.7546 (1.7648)	1.9623 (1.9853)	2.4674 (2.5308)
14	0.5007 (0.4753)	0.5966 (0.5859)	0.6501 (0.6453)	1.7096 (1.7186)	1.9018 (1.9222)	2.3643 (2.4206)
15	0.5161 (0.4923)	0.6088 (0.5989)	0.6608 (0.6564)	1.6710 (1.6791)	1.8502 (1.8685)	2.2775 (2.3278)
16	0.5298 (0.5076)	0.6198 (0.6107)	0.6705 (0.6664)	1.6374 (1.6448)	1.8055 (1.8220)	2.2031 (2.2485)
17	0.5421 (0.5212)	0.6298 (0.6213)	0.6792 (0.6755)	1.6080 (1.6147)	1.7665 (1.7815)	2.1388 (2.1800)
18	0.5532(0.5337)	0.6389 (0.6310)	0.6873 (0.6837)	1.5819 (1.5880)	1.7320 (1.7458)	2.0824 (2.1200)
19	0.5634 (0.5450)	0.6473 (0.6399)	0.6947 (0.6914)	1.5586 (1.5642)	1.7014 (1.7140)	2.0327 (2.0671)
20	0.5728 (0.5554)	0.6550 (0.6481)	0.7015 (0.6984)	1.5376 (1.5428)	1.6739 (1.6855)	1.9883 (2.0201)
25	0.6107 (0.5974)	0.6865 (0.6813)	0.7293 (0.7270)	1.4575 (1.4612)	1.5697 (1.5779)	1.8231 (1.8453)
30	0.6387 (0.6282)	0.7100 (0.7059)	0.7500 (0.7482)	1.4033 (1.4061)	1.4998 (1.5060)	1.7146 (1.7313)
35	0.6608 (0.6522)	0.7284 (0.7251)	0.7663 (0.7648)	1.3637 (1.3659)	1.4492 (1.4540)	1.6373 (1.6504)
40	0.6789 (0.6717)	0.7435 (0.7407)	0.7795 (0.7783)	1.3333 (1.3351)	1.4105 (1.4144)	1.5789 (1.5895)
50	0.7071 (0.7018)	0.7670 (0.7649)	0.8001 (0.7992)	1.2891 (1.2904)	1.3547 (1.3574)	1.4958 (1.5033)
60	0.7285 (0.7243)	0.7847 (0.7831)	0.8156 (0.8149)	1.2583 (1.2592)	1.3159 (1.3179)	1.4388 (1.4444)
70	0.7454 (0.7421)	0.7987 (0.7974)	0.8278 (0.8272)	1.2352 (1.2359)	1.2870 (1.2886)	1.3968 (1.4012)
80	0.7594 (0.7566)	0.8101 (0.8090)	0.8378 (0.8373)	1.2171 (1.2177)	1.2645 (1.2659)	1.3644 (1.3680)
90	0.7711 (0.7688)	0.8197 (0.8188)	0.8461 (0.8457)	1.2025 (1.2030)	1.2464 (1.2475)	1.3384 (1.3414)
100	0.7812 (0.7792)	0.8279 (0.8272)	0.8532 (0.8529)	1.1904 (1.1909)	1.2314 (1.2323)	1.3170 (1.3195)

from 3 to 50. The term under the radical sign in the approximate formula (7) could be negative for calculating some percentiles at small values of $n = 3$ and 4. For this reason, some approximate percentiles are not provided for $n = 3$ and 4 in Table 1. By comparing the approximate percentiles with the exact ones, we see that the approximate percentiles are reasonably accurate for calculating 90% and 95% CIs when sample sizes are 10 or more. The relative errors of the approximate percentiles are less than 5% when $n \geq 15$, and they can be used safely to construct CIs when $n \geq 30$.

4.2. Confidence limits for quantiles

A confidence limit for a quantile is commonly referred to as the tolerance limit. Specifically, an upper confidence limit for an upper quantile is called the upper tolerance limit, and a lower confidence limit for a lower quantile is called the lower tolerance limit. Noting that the p quantile of a two-parameter exponential distribution is given by $q_p = a - b\ln(1 - p)$, a pivotal quantity for q_p is given by $f_{n,c}$ with $c = -\ln(1 - p)$. Specifically, the pivotal quantity is given by

$$T_p = \frac{-2n\ln(1-p) - \chi_2^2}{\chi_{2n-2}^2}. \tag{11}$$

If $T_{p;q}$ denotes the $100q$ percentile of T_p, then

$$\widehat{a} + T_{p;1-\alpha}\widehat{b} \tag{12}$$

is a $1 - \alpha$ upper confidence limit for q_p, which in turn is a p-content – $(1 - \alpha)$ coverage upper tolerance limit for the exponential distribution. Similarly, it can be shown that $\widehat{a} + T_{1-p;\alpha}\widehat{b}$ is

Table 2. Exact and (approximate) factors for constructing one-sided tolerance limits.

	(0.90, 0.95)		(0.95, 0.95)		(0.99, 0.95)	
n	Lower	Upper	Lower	Upper	Lower	Upper
3	− 2.8184 (−3.0130)	16.7951 (16.6708)	− 3.1410 (−3.3632)	22.5991 (22.5104)	− 3.4052 (3.6405)	36.1376 (36.0846)
4	− 1.3587 (1.4238)	10.1308 (10.0659)	− 1.5350 (−1.5218)	13.4988 (13.4504)	− 1.6783 (−1.7677)	21.3469 (21.3165)
5	− 0.8538 (−0.8858)	7.7507 (7.7106)	− 0.9834 (−1.0254)	10.2736 (10.2431)	− 1.0883 (−1.1372)	16.1478 (16.1283)
6	− 0.6043 (−0.6229)	6.5441 (6.5164)	− 0.7119 (−0.7378)	8.6457 (8.6245)	− 0.7987 (−0.8301)	13.5365 (13.5228)
7	− 0.4570 (−0.4688)	5.8150 (5.7945)	− 0.5518 (−0.5694)	7.6650 (7.6492)	− 0.6283 (−0.6503)	11.9686 (11.9583)
8	− 0.3601 (−0.3680)	5.3255 (5.3095)	− 0.4468 (−0.4593)	7.0080 (6.9957)	− 0.5166 (−0.5330)	10.9209 (10.9129)
9	− 0.2917 (−0.2972)	4.9731 (4.9602)	− 0.3727 (−0.3821)	6.5359 (6.5260)	− 0.4379 (−0.4505)	10.1696 (10.1631)
10	− 0.2408 (−0.2448)	4.7065 (4.6959)	− 0.3177 (−0.3249)	6.1792 (6.1710)	− 0.3795 (−0.3896)	9.6029 (9.5975)
11	− 0.2016 (−0.2045)	4.4972 (4.4882)	− 0.2753 (−0.2810)	5.8995 (5.8926)	− 0.3344 (−0.3427)	9.1592 (9.1546)
12	− 0.1705 (−0.1726)	4.3281 (4.3204)	− 0.2416 (−0.2461)	5.6738 (5.6679)	− 0.2987 (−0.3056)	8.8015 (8.7976)
13	− 0.1451 (−0.1467)	4.1884 (4.1817)	− 0.2142 (−0.2179)	5.4875 (5.4823)	− 0.2697 (−0.2755)	8.5066 (8.5032)
14	− 0.1241 (−0.1253)	4.0707 (4.0649)	− 0.1915 (−0.1946)	5.3308 (5.3262)	− 0.2456 (−0.2506)	8.2587 (8.2557)
15	− 0.1064 (−0.1073)	3.9702 (3.9650)	− 0.1724 (−0.1749)	5.1969 (5.1928)	− 0.2253 (−0.2297)	8.0471 (8.0444)
16	− 0.0913 (−0.0919)	3.8832 (3.8785)	− 0.1560 (−0.1582)	5.0810 (5.0774)	− 0.2080 (−0.2118)	7.8642 (7.8618)
17	− 0.0782 (−0.0787)	3.8069 (3.8027)	− 0.1419 (−0.1438)	4.9797 (4.9764)	− 0.1931 (−0.1965)	7.7042 (7.7021)
18	− 0.0668 (−0.0672)	3.7396 (3.7357)	− 0.1297 (−0.1312)	4.8901 (4.8871)	− 0.1801 (−0.1831)	7.5631 (7.5611)
19	− 0.0568 (−0.0570)	3.6796 (3.6761)	− 0.1188 (−0.1201)	4.8104 (4.8076)	− 0.1686 (−0.1713)	7.4374 (7.4356)
20	− 0.0479 (−0.0481)	3.6257 (3.6225)	− 0.1092 (−0.1104)	4.7388 (4.7363)	− 0.1585 (−0.1609)	7.3247 (7.3230)
25	− 0.0152 (−0.0152)	3.4213 (3.4191)	− 0.0740 (−0.0746)	4.4677 (4.4659)	− 0.1211 (−0.1227)	6.8981 (6.8970)
30	0.0057 (0.0057)	3.2840 (3.2823)	− 0.0515 (−0.0518)	4.2857 (4.2844)	− 0.0974 (−0.0984)	6.6124 (6.6116)
35	0.02010 (0.0201)	3.1843 (3.1830)	− 0.0359 (−0.0361)	4.1538 (4.1528)	− 0.0809 (−0.0816)	6.4056 (6.4050)
40	0.0308 (0.0307)	3.1080 (3.1070)	− 0.0245 (−0.0246)	4.0531 (4.0523)	− 0.0688 (−0.0693)	6.2478 (6.2473)
50	0.0453 (0.0451)	2.9980 (2.9973)	− 0.0088 (−0.0088)	3.9079 (3.9073)	− 0.0522 (−0.0525)	6.0207 (6.0203)
60	0.0548 (0.0546)	2.9216 (2.9210)	0.0014 (0.0014)	3.8070 (3.8066)	− 0.0414 (−0.0416)	5.8632 (5.8629)
70	0.0615 (0.0612)	2.8647 (2.8642)	0.0086 (0.0086)	3.7321 (3.7317)	− 0.0338 (−0.0339)	5.7463 (5.7461)
80	0.0665 (0.0662)	2.8204 (2.8200)	0.0139 (0.0139)	3.6738 (3.6735)	− 0.0281 (−0.0282)	5.6554 (5.6552)
90	0.0703 (0.0700)	2.7847 (2.7844)	0.0180 (0.0180)	3.6268 (3.6266)	− 0.0238 (−0.0239)	5.5822 (5.5821)
100	0.0733 (0.0730)	2.7552 (2.7549)	0.0213 (0.0213)	3.5880 (3.5878)	− 0.0203 (−0.0204)	5.5218 (5.5217)

a p-content – $(1-\alpha)$ coverage lower tolerance limit for the exponential distribution. The percentile $T_{p;\alpha}$ can be obtained numerically using the cdf in (9) and a root finding method.

Approximation to the upper percentiles of T_p can be obtained by letting $X = -2n\ln(1-p) - \chi^2_2$, $Y = \chi^2_{2n-2}$, $\mu_x = -2n\ln(1-p) - 2$, $\mu_y = 2n-2$, $X_\alpha = -2n\ln(1-p) - \chi^2_{2;1-\alpha}$, and $Y_\alpha = \chi^2_{2n-2;\alpha}$ in (7). An approximate lower 100α percentile of T_{1-p} can be obtained as the negative of the approximate $100(1-\alpha)$ percentile of X/Y, where $X = 2n\ln(p) + \chi^2_2$ and $Y = \chi^2_{2n-2}$. These approximate percentiles can used as starting values to find the exact percentiles by an iterative numerical scheme.

We computed the exact percentiles of T_p in (11) for $(p, 1-\alpha) = (0.90, 0.95)$, $(0.95, 0.95)$, and $(0.99, 0.95)$, and for some selected sample sizes ranging from 3 to 100. These exact percentiles along with the approximate ones based on (7) are given in Table 2. Comparison of the approximate and exact percentiles clearly shows that the approximation is very satisfactory and the relative error is less than 2% when $n \geq 15$. In cases where percentiles are not reported here, the approximate percentiles can be safely used provided $n \geq 15$.

4.3. Survival probability

A confidence limit for a survival probability can be deduced from the tolerance limit in the preceding section. Let t denote the specified time at which we like to estimate the survival probability

$$S_t = P(X > t|a, b) = 1 - F(x|a, b) = \exp\left(-\frac{t-a}{b}\right).$$

Let $\widehat{a}_0$ and $\widehat{b}_0$ be the observed values of $\widehat{a}$ and $\widehat{b}$, respectively. The $1-\alpha$ lower confidence limit for $P(X > t)$ is the value of p for which the $(p, 1-\alpha)$ lower tolerance limit is equal to t (see Section 1.1.3 of Krishnamoorthy and Mathew 2009). After setting $(p, 1-\alpha)$ lower tolerance limit to t, it can be easily checked that p is determined by

$$P\left(\frac{\chi_2^2 + 2n\ln p}{\chi_{2n-2}^2} \le \frac{\widehat{a}_0 - t}{\widehat{b}_0}\right) = 1-\alpha \;\Leftrightarrow\; P\left(\exp\left\{-\frac{1}{2n}A\right\} \ge p\right) = 1-\alpha,$$

where

$$A = \left(\frac{t-\widehat{a}_0}{\widehat{b}_0}\right)\chi_{2n-2}^2 + \chi_2^2. \tag{13}$$

Thus, p should be the α quantile of $\exp\{-\frac{1}{2n}A\}$ which in turn is a $100(1-\alpha)\%$ lower confidence limit of $P(X > t)$. Finally, the $1-2\alpha$ CI for $P(X > t)$ is expressed as

$$\left(\exp\left\{-\frac{1}{2n}A_{1-\alpha}\right\},\ \exp\left\{-\frac{1}{2n}A_{\alpha}\right\}\right), \tag{14}$$

where A_q is the q quantile of A.

The cdf of A can be expressed as follows. Let $w_0 = (t-\widehat{a}_0)/\widehat{b}_0$ so that $A = w_0\chi_{2n-2}^2 + \chi_2^2$. For a given $t \ge \widehat{a}_0$, following the derivation the cdf f_{nc} in Appendix, we can find

$$P(A \le x) = \begin{cases} 1 - e^{-x/2}, & w_0 = 0, \\ F_{2n-2}(\frac{x}{w_0}) - e^{-\frac{x}{2w_0}}\left[\sum_{m=0}^{N}\frac{(1-w_0)^m}{(m+n-1)!}\left(\frac{x}{2w_0}\right)^{m+n-1}\right] + \text{error}, & \frac{1}{2} \le w_0 \le \frac{3}{2}, \\ F_{2n-2}(\frac{x}{w_0}) - \frac{e^{-x/2}}{(1-w_0)^{n-1}}\left(1 - e^{-\frac{x(1-w_0)}{2w_0}}\sum_{j=0}^{n-2}\frac{1}{j!}\left(\frac{x(1-w_0)}{2w_0}\right)^j\right), & 0 < w_0 < \frac{1}{2} \text{ or } w_0 > \frac{3}{2}, \end{cases} \tag{15}$$

where $F_m(x)$ denotes the Chi-square cdf with df $= m$, N is the smallest positive integer equal to or greater than $\max\{1, (e^2x - n), \ln(\frac{1}{\sqrt{2\pi\epsilon}}) - n\}$, and the "error" is less than the specified error tolerance ϵ.

A convenient approximation to the percentiles of A can be obtained using the MNA in (6). Noting that for a given $(t, \widehat{a}_0, \widehat{b}_0)$, A is a linear combination of independent Chi-square random variables, an approximate 100α percentile of A is given by

$$A_\alpha \simeq w_0(2n-2) + 2 - \left[w_0^2(2n-2-L^*)^2 + \left(2-\chi_{2;\alpha}^2\right)\right]^{\frac{1}{2}}, \tag{16}$$

where $L^* = \chi_{2n-2;\alpha}^2$ if $w_0 > 0$ and is $\chi_{2n-2;1-\alpha}^2$ otherwise. The approximate $1-\alpha$ upper confidence limit for A is expressed as

$$A_{1-\alpha} \simeq w_0(2n-2) + 2 + \left[w_0^2(2n-2-U^*)^2 + \left(2-\chi_{2;1-\alpha}^2\right)\right]^{\frac{1}{2}}, \tag{17}$$

where $U^* = \chi_{2n-2;1-\alpha}^2$ if $w_0 > 0$, and is $\chi_{2n-2;\alpha}^2$ otherwise.

5. Confidence intervals: Two-sample case

We shall describe an interval estimation method for the difference between two location parameters and for the difference between two means based on the generalized variable approach. Let $\widehat{a}_{i0}$ and $\widehat{b}_{i0}$ be the observed values of $\widehat{a}_i$ and $\widehat{b}_i$, respectively. Based on the distributional results (3), generalized pivotal quantities (GPQs) for a_i and b_i are

$$G_{a_i} = \widehat{a}_{i0} - \frac{\chi_2^2}{\chi_{2n-2}^2}\widehat{b}_{i0} \quad \text{and } G_{b_i} = \frac{2n_i\widehat{b}_{i0}}{\chi_{2n-2}^2}, \tag{18}$$

respectively. Details on finding GPQs for the parameters of a two-parameter exponential distribution can be found in Roy and Mathew (2005) and Krishnamoorthy, Mukherjee, and Guo (2007).

5.1. CIs for the difference between two location parameters

Generalized CI

The GPQ for the difference $a_1 - a_2$, denoted by $G_{a_1-a_2}$, is obtained by substitution as

$$\begin{aligned} G_{a_1-a_2} &= G_{a_1} - G_{a_2} \\ &= \widehat{a}_{10} - \frac{X_1}{Y_1}\widehat{b}_{10} - \widehat{a}_{20} + \frac{X_2}{Y_2}\widehat{b}_{20} \\ &= \widehat{a}_{10} - \widehat{a}_{20} - (c_1F_{2,2n_1-2} - c_2F_{2,2n_2-2}), \end{aligned} \tag{19}$$

where the random variables X_1, X_2, Y_1, and Y_2 are mutually independent with $X_i \sim \chi_2^2$, $Y_i \sim \chi_{2n_i-2}^2$, and $c_i = \widehat{b}_{i0}/(n_i-1)$, $i = 1, 2$. Furthermore, $F_{m,n}$ is an F random variable with the numerator dfs m and the denominator df n, and those in (19) are independent. Note that, for a given $(\widehat{a}_{10}, \widehat{b}_{10}, \widehat{a}_{20}, \widehat{b}_{20})$, the distribution of the GPQ in (19) does not depend on any unknown parameters, and so they can be estimated using Monte Carlo simulation. Specifically, letting $T_{n_1,n_2} = c_1F_{2,2n_1-2} - c_2F_{2,2n_2-2}$, we see from (19) that a CI for $a_1 - a_2$ is determined by the percentiles of T_{n_1,n_2}.

A closed-form approximate CI for $a_1 - a_2$ can also be obtained by approximating the percentiles of T_{n_1,n_2} using the approximation in (6). To find the approximation, note that $u_i = E(F_{2,2n_i-2}) = \frac{n_i-1}{n_i-2}$, $n_i > 2$, $i = 1, 2$. Using these expectations in (6), we find

$$T_{n_1,n_2;\alpha} \approx \begin{cases} c_1u_1 - c_2u_2 - \sqrt{c_1^2(u_1 - F_{2,2n_1-2;\alpha})^2 + c_2^2(u_2 - F_{2,2n_2-2;1-\alpha})^2}, & 0 < \alpha < .5, \\ c_1u_1 - c_2u_2 + \sqrt{c_1^2(u_1 - F_{2,2n_1-2;\alpha})^2 + c_2^2(u_2 - F_{2,2n_2-2;1-\alpha})^2}, & 0.5 < \alpha < 1, \end{cases} \tag{20}$$

where $c_i = \widehat{b}_{i0}/(n_i-1)$, $i = 1, 2$. The approximate $1 - 2\alpha$ CI for $a_1 - a_2$ is given by

$$(\widehat{a}_{10} - \widehat{a}_{20} - T_{n_1n_2;1-\alpha},\ \widehat{a}_{10} - \widehat{a}_{20} - T_{n_1n_2;\alpha}). \tag{21}$$

Alternatively, one could also approximate the percentiles of T_{n_1,n_2} by replacing the mean u_i in (20) with the median $M_i = F_{2,2n_i-2;.5}$, $i = 1, 2$. Our simulation studies (not reported here) indicated that the approximation with medians produces CIs for $a_1 - a_2$ with better coverage probabilities, and so in the sequel we shall consider only the CI with $u_i = F_{2,2n_i-2;.5}$, $i = 1, 2$.

Kharrati-Kopaei CI

Kharrati-Kopaei (2015) has proposed a simple method for finding simultaneous CIs for successive differences of $a_{i+1} - a_i$, $i = 1, \ldots, k-1$, where $a_1, \ldots, a_k$ are location parameters of k independent exponential distributions. To describe his approach, let $X_{i1}, \ldots, X_{in_i}$ be a sample from the exponential distribution with the location parameter a_i and the scale parameter b_i, $i = 1, \ldots, k$. Let $\widehat{a}_i = X_{(i1)} = \min\{X_{i1}, \ldots, X_{in_i}\}$, $\widehat{C}_i = \widehat{b}_i/(n_i-1)$, and $d_i = F_{2,2n_i-2;(1-\alpha)^{1/k}}$, $i = 1, \ldots, k$. Then

$$(\widehat{a}_{i+1} - \widehat{a}_i - \widehat{C}_{i+1}d_{i+1}, \widehat{a}_{i+1} - \widehat{a}_i - \widehat{C}_id_i) \quad i = 1, \ldots, k-1, \tag{22}$$

are simultaneous CIs for $a_{i+1} - a_i$ with confidence at least $1-\alpha$. Letting $k=2$, we have a CI

$$(\widehat{a}_2 - \widehat{a}_1 - \widehat{C}_2 d_2,\ \widehat{a}_2 - \widehat{a}_1 + \widehat{C}_1 d_1)$$

for $a_2 - a_1$, or equivalently, we have a CI

$$(\widehat{a}_1 - \widehat{a}_2 - \widehat{C}_1 d_1,\ \widehat{a}_1 - \widehat{a}_2 + \widehat{C}_2 d_2) \tag{23}$$

for $a_1 - a_2$ with confidence at least $1-\alpha$.

5.2. CIs for the difference between two means

Let $(\widehat{a}_{i0}, \widehat{b}_{i0})$ be an observed value of $(\widehat{a}_i, \widehat{b}_i)$, $i = 1, 2$. A GPQ for the mean $\mu_i = a_i + b_i$ can be obtained by substitution as

$$G_{\mu_i} = G_{a_i} + G_{b_i} = \widehat{a}_{i0} + \left(\frac{2n_i - \chi_2^2}{\chi_{2n_i-2}^2}\right)\widehat{b}_{i0},$$

where the Chi-square random variables are independent. The lower and upper α quantiles of G_{μ_i} form a $1-2\alpha$ CI for μ_i, which is the same as the exact CI in (10). A GPQ for $\mu_1 - \mu_2$ is obtained as

$$\begin{aligned} G_{\mu_1} - G_{\mu_2} &= \widehat{a}_{10} - \widehat{a}_{20} + \widehat{b}_{10}\left(\frac{2n_1 - \chi_2^2}{\chi_{2n_1-2}^2}\right) - \widehat{b}_{20}\left(\frac{2n_2 - \chi_2^2}{\chi_{2n_2-2}^2}\right) \\ &= \widehat{a}_{10} - \widehat{a}_{20} + \widehat{b}_{10} f_{n_1,1} - \widehat{b}_{20} f_{n_2,1}, \end{aligned} \tag{24}$$

where $f_{n_i,1}$ is defined in (4). For a given $(\widehat{a}_{10}, \widehat{b}_{10}, \widehat{a}_{20}, \widehat{b}_{20})$, the distribution of $G_{\mu_1} - G_{\mu_2}$ does not depend on any unknown parameters, and so they can be estimated by Monte Carlo simulation. The lower $\alpha/2$ quantile and the upper $\alpha/2$ quantile of $G_{\mu_1} - G_{\mu_2}$ form a $1-\alpha$ CI for $\mu_1 - \mu_2$.

The percentiles of $G_{\mu_1} - G_{\mu_2}$ can also be approximated using the MNA as follows. It follows from (24) that to find a CI for $\mu_1 - \mu_2$, it is enough to find percentiles of $Q_{n_1,n_2} = \widehat{b}_{10} f_{n_1,1} - \widehat{b}_{20} f_{n_2,1}$, which can be approximated using the approximation in (6) as follows. Let $u_i = E(f_{n_i,1}) = (n_i - 1)/(n_i - 2)$, $i = 1, 2$. Then

$$Q_{n_1,n_2;\alpha} \approx \begin{cases} \widehat{b}_{10}u_1 - \widehat{b}_{20}u_2 - \sqrt{\widehat{b}_{10}^2(u_1 - f_{n_1,1;\alpha})^2 + \widehat{b}_{20}^2(u_2 - f_{n_2,1;1-\alpha})^2}, & 0 < \alpha < 0.5, \\ \widehat{b}_{10}u_1 - \widehat{b}_{20}u_2 + \sqrt{\widehat{b}_{10}^2(u_1 - f_{n_1,1;1-\alpha})^2 + \widehat{b}_{20}^2(u_2 - f_{n_2,1;\alpha})^2}, & 0.5 < \alpha < 1. \end{cases} \tag{25}$$

One could also use the median of $f_{n_i,1}$ instead of the mean u_i, $i = 1, 2$, in the above approximation. Our preliminary numerical studies indicated that the median of $f_{n_i,1}$ is very close to $E(f_{n_i,1})$, and so the above approximations with the mean or the median should be very similar.

Using the approximations in (25), we find a $1-2\alpha$ CI for $\mu_1 - \mu_2$ as

$$(\widehat{a}_{10} - \widehat{a}_{20} + Q_{n_1,n_2;\alpha}, \widehat{a}_{10} - \widehat{a}_{20} + Q_{n_1,n_2;1-\alpha}). \tag{26}$$

The above CI is very satisfactory in terms of coverage probabilities as suggested by simulation studies below.

5.3. Coverage studies

To study the coverage properties of the approximate CI (21) and the Kharrati-Kopaei CI in (23) for $a_1 - a_2$, we estimated the coverage probabilities and expected widths of 95% CIs for some small-to-moderate sample sizes and presented them in Table 3. Note that both CIs are location–scale equivariant, and so without loss of generality we can assume $a_1 = a_2 = 0$, $b_1 = 1$, and $0 < b_2 < 1$ for comparing the CIs in terms of coverage probabilities and expected widths. Coverage probabilities and expected widths were estimated for [1] the CI based on the approximate percentiles $T_{n_1,n_2;\alpha}$ with $u_i = F_{2,2n_i-2;.5}$ and [2] Kharrati-Kopaei's CI. The coverage probabilities in Table 3 clearly indicate that our CI [1] controls the coverage probabilities very close to the nominal level 0.95, except for the case $n_1 = 3$ and $n_2 = 3$, where the coverage probabilities are a little above 0.95. The CI [2] is conservative with coverage probabilities being close to 0.975 for all the cases considered, and so it is expected to be wider than that of CI [1]. The percentage of relative reduction of average length of CI [1] over CI [2] is increasing with decreasing disparity between b_1 and b_2. For example, when $n_1 = n_2 = 3$ and $(b_1, b_2) = (0.001, 1)$, the average lengths of both CIs are practically the same, whereas at $n_1 = n_2 = 30$ and $(b_1, b_2) = (1, 1)$, the percentage relative reduction in expected width is $[(0.26 - 0.22)/0.26] \times 100 = 15.4\%$. On the basis of the expected widths reported in Table 3, the relative reduction in the average length of our CI over the Kharrati-Kopaei CI is ranging from 0% to 15%. A reviewer has noted that the Kharrati-Kopaei CI is slightly shorter than our CI in some extreme cases. For example, when $(n_1, n_2) = (3, 3), (4, 3)$, and $(b_1, b_2) = (0.0001, 1), (0.00005, 1), (0.00002, 1)$, the CI [2] is slightly shorter than CI [1]. Our calculation for these cases showed that the relative reduction in the average length of CI [2] over the CI [1] is about 0.5%. Thus, the CI [1] is not shorter than the CI [2] for all paramter values. However, as the reduction in the average length is minute, we can say that both CIs are comparable in such unrealistic cases. Since the improvement of CI [1] over CI [2] outweighs the diminishment, on an overall basis, our new CI is preferable for practical applications.

We also estimated the coverage probabilities of the CI based on the GPQ in (24) and the approximate CI in (26) for the difference between two means for small-to-moderate sample sizes, and we reported them in Table 4. The coverage probability of the CI based on the GPQ is estimated as follows. Assuming some values for parameters, we first generated 10,000 statistics $(\widehat{a}_1, \widehat{b}_1, \widehat{a}_2, \widehat{b}_2)$ based on some assumed sample sizes (n_1, n_2). For each set of statistics, we estimated the 95% CI for the mean difference based on simulation consisting of

Table 3. Coverage probabilities and (expected widths) of 95% CIs for the difference between two location parameters. $a_1 = a_2 = 0; b_1 = 1$

	(n_1, n_2)							
	(3, 3)		(5, 5)		(10, 15)		(30, 30)	
b_2	[1]	[2]	[1]	[2]	[1]	[2]	[1]	[2]
0.001	0.951 (3.55)	0.975 (3.53)	0.950 (1.21)	0.975 (1.21)	0.950 (0.28)	0.975 (0.28)	0.949 (0.13)	0.975 (0.13)
0.01	0.950 (3.55)	0.975 (3.56)	0.950 (1.21)	0.974 (1.22)	0.947 (0.28)	0.974 (0.29)	0.948 (0.13)	0.974 (0.13)
0.1	0.959 (3.73)	0.975 (3.87)	0.953 (1.25)	0.975 (1.33)	0.950 (0.30)	0.975 (0.33)	0.950 (0.13)	0.974 (0.14)
0.2	0.958 (4.02)	0.975 (4.22)	0.954 (1.33)	0.975 (1.45)	0.951 (0.47)	0.975 (0.51)	0.951 (0.14)	0.975 (0.16)
0.3	0.957 (4.32)	0.974 (4.57)	0.952 (1.42)	0.974 (1.57)	0.951 (0.49)	0.974 (0.54)	0.951 (0.15)	0.975 (0.17)
0.5	0.956 (4.95)	0.974 (5.27)	0.953 (1.62)	0.974 (1.81)	0.953 (0.52)	0.974 (0.60)	0.951 (0.17)	0.974 (0.20)
0.7	0.957 (5.62)	0.975 (5.99)	0.954 (1.83)	0.975 (2.05)	0.952 (0.57)	0.973 (0.65)	0.948 (0.19)	0.975 (0.22)
0.8	0.957 (5.95)	0.975 (6.35)	0.953 (1.93)	0.974 (2.17)	0.952 (0.59)	0.975(0.68)	0.948 (0.20)	0.975 (0.24)
1.0	0.956 (6.59)	0.975(7.03)	0.954 (2.15)	0.975 (2.41)	0.950 (0.63)	0.975 (0.73)	0.949 (0.22)	0.974 (0.26)

Note: [1] = CI (21) based on $T_{n_1,n_2;\alpha}$ with $u_i = F_{2,2n_i-2,1;.5}$; [2] = CI (23) due to Kharrati-Kopaei (2015)

Table 4. Coverage probabilities of $1-\alpha$ CIs for the difference between two means. $a_1 = a_2 = 0$; $b_1 = 1$; $1-\alpha = .95$

	(n_1, n_2)					
	(3, 3)		(5, 5)		(5, 10)	
b_2	1	2	1	2	1	2
0.1	0.960 (6.51)	0.955 (6.45)	0.952 (2.91)	0.953 (2.90)	0.951 (2.87)	0.951 (2.87)
0.2	0.969 (6.92)	0.961 (6.69)	0.957 (2.99)	0.954 (2.98)	0.948 (2.91)	0.952 (2.89)
0.3	0.972 (7.33)	0.967 (7.10)	0.959 (3.15)	0.957 (3.13)	0.956 (2.95)	0.953 (2.93)
0.5	0.979 (8.16)	0.974 (7.94)	0.963 (3.52)	0.959 (3.46)	0.956 (3.11)	0.955 (3.04)
0.8	0.985 (9.21)	0.981 (8.85)	0.966 (3.93)	0.963 (3.85)	0.958 (3.24)	0.956 (3.20)
0.9	0.987 (9.69)	0.983 (9.37)	0.971 (4.18)	0.963 (4.06)	0.957 (3.32)	0.958 (3.28)
1	0.989 (10.8)	0.984 (10.3)	0.968 (4.60)	0.964 (4.49)	0.959 (3.49)	0.958 (3.46)
	(5, 15)		(10, 10)		(15, 20)	
b_2	1	2	1	2	1	2
0.1	0.954 (2.86)	0.950 (2.86)	0.949 (1.54)	0.949 (1.54)	0.948 (1.16)	0.949 (1.16)
0.2	0.952 (2.88)	0.950 (2.88)	0.952 (1.58)	0.950 (1.58)	0.951 (1.18)	0.950 (1.17)
0.3	0.949 (2.90)	0.951 (2.90)	0.956 (1.63)	0.952 (1.63)	0.951 (1.21)	0.950 (1.20)
0.5	0.956 (2.99)	0.955 (2.96)	0.955 (1.81)	0.952 (1.78)	0.952 (1.29)	0.951 (1.28)
0.8	0.956 (3.08)	0.955 (3.05)	0.953 (1.99)	0.951 (1.97)	0.950 (1.38)	0.949 (1.38)
0.9	0.960 (3.13)	0.956 (3.12)	0.948 (2.11)	0.952 (2.08)	0.953 (1.45)	0.950 (1.44)
1	0.957 (3.27)	0.956 (3.22)	0.953 (2.34)	0.952 (2.30)	0.950 (1.56)	0.950 (1.56)

1 = CI based on the GPQ (24); 2 = CI based on (26)

10,000 runs. The proportion of the 10,000 CIs that include the assumed mean difference is an estimate of the coverage probability when the nominal level is 0.95. These estimated coverage probabilities in Table 4 indicate that both CIs for the difference between two means could be slightly conservative for very small sample sizes such as five or less. The coverage probabilities of the CI based on the GPQ appear to be slightly larger than those of the approximate CI for $(n_1, n_2) = (3, 3)$ and $(5, 5)$. In general, we see that both CIs control the coverage probabilities very close to the nominal levels with similar expected widths for sample sizes of five or more. Even though both CIs are reasonably accurate when both sample sizes are five or more, we recommend the approximate one as it is in closed form and does not require simulation.

We also estimated the coverage probabilities and expected widths of 90% and 99 % CIs for the difference between location parameters and for the difference between two means. Since the performance and comparison of CIs are very similar to the 95% CIs, they are not reported here.

6. Examples

Example 1. To illustrate the methods in the preceding sections, we shall use the failure mileage data on 19 military carriers given in Grubbs (1971). The data are reproduced here in Table 5. Engelhardt and Bain (1978), and many others considered this data for illustrating the methods for two-parameter exponential distribution. The MLEs based on the data are $\widehat{a} = X_{(1)} = 162$ and $\widehat{b} = 835.21$.

Table 5. Failure mileage of 19 military carriers.

162	200	271	302	393	508	539	629	706	777
884	1008	1101	1182	1463	1603	1984	2355	2880	

To compute the 95% CIs for the mean, the exact value of $(f_{19,1;.025}, f_{19,1;.975})$ is (0.6473, 1.7014), and the approximate one is (0.6399, 1.7140). The exact CI for the mean is (**702.6**, **1583.0**), and the approximate one is $(\widehat{a} + 0.6399\widehat{b},\ \widehat{a} + 1.7014\widehat{b}) = $ (**696.5**, **1593.6**). These two CIs are very similar, not appreciably different.

In this type of problem, it is of interest to find the lower tolerance limit to judge the minimum life span of a product. So we shall compute a (0.95, 0.95) lower tolerance limit for failure mileage distribution. The exact (0.95, 0.95) factor for finding the lower tolerance limit is −0.1188, and the tolerance limit is $162 - 0.1188 \times 835.2 = 62.78$. This means that at least 95% of military carriers will last **62.78** units of miles with confidence 95%. The approximate factor is −0.1201, and the tolerance limit is $162 - 0.1201 \times 835.21 =$ **62.69**, which is very close to the exact one.

Suppose it is desired to find a 95% lower confidence limit for the probability that a military carrier last 300 or more units of miles, that is, $P(X > 300)$, where X represents the failure milage of a military carrier. Recall that $\widehat{a}_0 = 162$ and $\widehat{b}_0 = 835.21$. The exact 95th percentile of A defined in (13) is 12.495, and the exact 95% lower confidence limit for $P(X > 300)$ is **0.720**. This means that at least 72% of military carriers work 300 units of miles or more with confidence 95%. To compute the approximate lower confidence limit based on (16), we found $U^* = \chi^2_{36;.95} = 50.998$, $\chi^2_{2;.95} = 5.991$, $w_0 = 0.1652$, and $A_{.95} = 12.646$. So the 95% lower confidence limit for $P(X > 300)$ is $\exp(-12.646/2/19) =$ **0.717**, which is very close to the exact one.

Example 2. To illustrate the two-sample methods, we shall use the data given in Table 2 of Kharrati-Kopaei (2015). The data represent the survival days of patients with inoperable lung cancer. The data were categorized into four groups depending on the historical type of patients' tumor, namely, squamous, small, adeno, and large. We shall consider the first two groups with the data given in Table 6 for the purpose of illustration. For these samples, note that $n_1 = n_2 = 9$. The MLEs are $\widehat{a}_1 = 8$, $\widehat{b}_1 = 43$, $\widehat{a}_2 = 13$, and $\widehat{b}_2 = 9.111$. The 95% CI for $a_1 - a_2$ based on (21) with 1,000,000 simulation runs is (−29.00, −2.10). To find the approximate 95% CI for the difference between the location parameters, we found $T_{15,20;.025}$ using (20) as −2.80 and $T_{15,20;.975}$ as 24.38. To calculate these quantiles, we used $u_1 = u_2 = F_{2,2n_1-2;.5} = 0.7241$. So the 95% CI for $a_1 - a_2$ is

$$(\widehat{a}_1 - \widehat{a}_2 - 24.38,\ \widehat{a}_1 - \widehat{a}_2 + 2.80) = (-29.38, -2.20).$$

To find the Kharrati-Kopaei CI in (23), we calculated $C_1 = 5.375, C_2 = 1.139$, and $d_1 = d_2 = 4.666$. Using these quantities in (23), we obtain the CI as (−30.08, .315). Notice that this CI is wider than the approximate one, because Kharrati-Kopaei's CI is conservative. Furthermore, this CI indicates that the difference $a_1 - a_2$ is not significant, whereas the approximate one indicates significant.

To find the 95% CI for the difference between the means, we evaluated

$$u_1 = u_2 = \frac{n_1 - 1}{n_2 - 1} = 1.1429,\ f_{n_1,1;.025} = f_{n_2,1;.025} = .5027,\ f_{n_1,1;.975} = f_{n_2,1;.975} = 2.3693.$$

Table 6. Survival days of patients based on their tumor types.

1. Squamous	72	10	81	110	100	42	8	25	11
2. Small	30	13	23	16	21	18	20	27	31

Furthermore, $Q_{n_1,n_2;.025} = 9.022$ and $Q_{n_1,n_2;.975} = 91.789$. Thus, the approximate 95% CI for the mean difference is

$$(\widehat{a}_1 - \widehat{a}_2 - 9.022, \widehat{a}_1 - \widehat{a}_2 + 91.789) = (4.02, 86.79).$$

The 95% CI based on (24) with 1,000,000 simulation runs is (4.01, 86.83), which is practically the same as the approximate one. Both CIs indicate that the mean difference of these two groups is significantly different.

The R functions that were used to find the confidence limits are provided as a supplementary file at the journal's web site.

7. Conclusions

In this article, we have derived the distribution function of a pivotal quantity whose percentiles can be used to find CIs for the mean and to construct one-sided tolerance limits. An exact method of finding confidence limits for a survival probability is also outlined. A convenient approximation has been provided for each of the problems considered. In order to aid practitioners to apply the proposed methods, table values are given to find CIs for the mean and quantiles. Our simulation studies clearly indicate that the approximate CIs for comparing two location parameters and for comparing means are accurate, and they can be used safely in practical applications. Given the MLEs of the location and scale parameters, only the Chi-square percentiles are required to compute the CIs, and so the methods are simple to use in applications. To aid practitioners to use the proposed methods, R codes are provided in a supplementary file that can be obtained from the journal's web site or from the homepage www.ucs.louisiana.edu/∼kxk4695/ of the first author.

The approximate solutions in the preceding sections can be readily extended to the problems involving type II censored (failure censored) samples. Noting that the MLEs in the type II censored case have similar Chi-square distributions (see Theorem 4.5.1 of Lawless 2003), one can easily obtain a pivotal quantity for $a + cb$, where c is a known positive constant, from which pivotal quantities for the special cases (mean and quantile) can be readily deduced. Furthermore, the results for the two-sample problems can also be extended in a straightforward manner.

There are a few articles that proposed the methods for finding simultaneous CIs for pairwise differences of means or location parameters; for example, see Kharrati-Kopaei (2015) and Li, Song and Shi (2015) and the references therein. Our proposed approach for the two-sample problems can be extended to find simultaneous CIs for pairwise differences of means or location parameters by using the Bonferroni method or other multiple comparison method. We plan to work on how such simultaneous CIs compare with those already proposed in the literature.

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Appendix

Recall that the pdf of a χ^2_{2m} random variable with positive integer m can be expressed as

$$f_{2m}(x) = \frac{1}{2^m \Gamma(m)} x^{m-1} e^{-\frac{x}{2}}.$$

and the cdf of χ^2_{2m} is

$$F_{2m}(x) = \int_0^x f_{2m}(x)dx = 1 - e^{-\frac{x}{2}} \sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{x}{2}\right)^j = e^{-\frac{x}{2}} \sum_{j=m}^{\infty} \frac{1}{j!}\left(\frac{x}{2}\right)^j, \quad x \geq 0. \tag{A1}$$

For $t > 0$ and $t \neq 1$, the cdf of $f_{n,c}$ can be evaluated as

$$
\begin{aligned}
F_{n,c}(t) &= P\left(\frac{2nc-\chi_2^2}{\chi_{2n-2}^2} \le t\right) = P\left(\chi_2^2 + t\chi_{2n-2}^2 \ge 2nc\right) \\
&= \int_0^\infty \left(\int_{\max(0,2nc-ty)}^\infty f_2(x)dx\right) f_{2n-2}(y)dy \\
&= \int_{\frac{2nc}{t}}^\infty \left(\int_0^\infty f_2(x)dx\right) f_{2n-2}(y)dy + \int_0^{\frac{2nc}{t}} \left(\int_{2nc-ty}^\infty f_2(x)dx\right) f_{2n-2}(y)dy \\
&= \int_{\frac{2nc}{t}}^\infty f_{2n-2}(y)dy + \int_0^{\frac{2nc}{t}} [1-F_2(2nc-ty)]f_{2n-2}(y)dy \\
&= 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + e^{-nc}\int_0^{\frac{2nc}{t}} \frac{1}{2^{n-1}\Gamma(n-1)} y^{n-2} e^{-\frac{(1-t)y}{2}} dy \\
&= 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + \frac{e^{-nc}}{(1-t)^{n-1}}\int_0^{\frac{2nc(1-t)}{t}} \frac{1}{(n-2)!} s^{n-2} e^{-s} ds.
\end{aligned}
$$

Since

$$
\frac{1}{m!}\int_0^z s^m e^{-s} ds = 1 - e^{-z}\sum_{j=0}^m \frac{1}{j!} z^j,
$$

we have

$$
\begin{aligned}
P\left(\frac{2nc-\chi_2^2}{\chi_{2n-2}^2} \le t\right) &= 1 - F_{2n-2}\left(\frac{2nc}{t}\right) \\
&\quad + \frac{e^{-nc}}{(1-t)^{n-1}}\left(1 - e^{-\frac{nc(1-t)}{t}}\sum_{j=0}^{n-2}\frac{1}{j!}\left(\frac{nc(1-t)}{t}\right)^j\right). \qquad \text{(A2)}
\end{aligned}
$$

We shall now express the $F_{n,c}(t)$ for various values of t.

For $t = 1$, it can be easily checked that

$$
P\left(\frac{2nc-\chi_2^2}{\chi_{2n-2}^2} \le t\right) = P(\chi_{2n}^2 \ge 2nc) = 1 - F_{2n}(2nc).
$$

For $t \le 0$, we see that

$$
\begin{aligned}
P\left(\frac{2nc-\chi_2^2}{\chi_{2n-2}^2} \le t\right) &= P\left(\chi_2^2 + t\chi_{2n-2}^2 \ge 2nc\right) \\
&= \int_0^\infty \left(\int_{\max(0,2nc-ty)}^\infty f_2(x)dx\right) f_{2n-2}(y)dy \\
&= \int_0^\infty \left(\int_{2nc-ty}^\infty f_2(x)dx\right) f_{2n-2}(y)dy \\
&= \int_0^\infty [1-F_2(2nc-ty)]f_{2n-2}(y)dy \\
&= e^{-nc}\int_0^\infty \frac{1}{2^{n-1}\Gamma(n-1)} y^{n-2} e^{-\frac{(1-t)y}{2}} dy
\end{aligned}
$$

$$= \frac{e^{-nc}}{(1-t)^{n-1}} \int_0^\infty \frac{1}{2^{n-1}\Gamma(n-1)} s^{n-2} e^{-\frac{s}{2}} ds$$
$$= \frac{e^{-nc}}{(1-t)^{n-1}}.$$

Evaluation of the cdf when t is near unity

Since

$$1 - e^{-z}\sum_{j=0}^{n-2} \frac{1}{j!} z^j = e^{-z} \sum_{j=n-1}^{\infty} \frac{1}{j!} z^j,$$

it follows from (A2) that

$$\begin{aligned} P\left(\frac{2nc-\chi_2^2}{\chi_{2n-2}^2} \le t\right) &= 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + \frac{e^{-nc}}{(1-t)^{n-1}} \cdot e^{-\frac{nc(1-t)}{t}} \sum_{j=n-1}^{\infty} \frac{1}{j!}\left(\frac{nc(1-t)}{t}\right)^j \\ &= 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + e^{-\frac{nc}{t}} \sum_{j=n-1}^{\infty} \frac{1}{j!}\left(\frac{nc}{t}\right)^j (1-t)^{j-(n-1)} \\ &= 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + e^{-\frac{nc}{t}} \sum_{m=0}^{\infty} \frac{1}{(m+n-1)!}\left(\frac{nc}{t}\right)^{m+n-1} (1-t)^m \\ &= 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + e^{-\frac{nc}{t}} \sum_{m=0}^{N} \frac{1}{(m+n-1)!}\left(\frac{nc}{t}\right)^{m+n-1} (1-t)^m + E_N, \end{aligned}$$

where

$$E_N = e^{-\frac{nc}{t}} \sum_{m=N+1}^{\infty} \frac{1}{(m+n-1)!}\left(\frac{nc}{t}\right)^{m+n-1} (1-t)^m.$$

For $\frac{1}{2} \le t \le \frac{3}{2}$, or equivalently, $|1-t| \le \frac{1}{2}$, we have

$$\begin{aligned} |E_N| &\le e^{-\frac{nc}{t}} |1-t|^{N+1} \sum_{m=N+1}^{\infty} \frac{1}{(m+n-1)!}\left(\frac{nc}{t}\right)^{m+n-1} \\ &= e^{-\frac{nc}{t}} |1-t|^{N+1} \sum_{k=N+n}^{\infty} \frac{1}{k!}\left(\frac{nc}{t}\right)^k. \end{aligned}$$

By Taylor's remainder theorem, we see that

$$\sum_{k=N+n}^{\infty} \frac{1}{k!}\left(\frac{nc}{t}\right)^k = e^{\frac{nc}{t}} - \sum_{k=0}^{N+n-1} \frac{1}{k!}\left(\frac{nc}{t}\right)^k = \frac{e^{\xi}}{(N+n)!}\left(\frac{nc}{t}\right)^{N+n}$$

for some positive number ξ between 0 and $\frac{nc}{t}$. Thus, for $\frac{1}{2} \le t \le \frac{3}{2}$,

$$\begin{aligned} |E_N| &\le e^{-\frac{nc}{t}} |1-t|^{N+1} \frac{e^{\xi}}{(N+n)!}\left(\frac{nc}{t}\right)^{N+n} \\ &\le \frac{\left(\frac{nc}{t}\right)^{N+n}}{(N+n)!} \\ &\le \frac{(2nc)^{N+n}}{(N+n)!}. \end{aligned}$$

Using the inequality (https://en.wikipedia.org/wiki/Stirling%27s_approximation) that

$$\sqrt{2\pi}\, k^{k+\frac{1}{2}} e^{-k} \le k!,$$

we can further simplify the upper bound of $|E_N|$ as

$$\begin{aligned}|E_N| &\le \frac{(2nc)^{N+n} e^{N+n}}{\sqrt{2\pi}\,(N+n)^{N+n+\frac{1}{2}}}\\ &= \frac{1}{\sqrt{2\pi}} \exp\left((N+n)\,(1+\ln(2nc)) - \left(N+n+\frac{1}{2}\right)\ln(N+n)\right)\\ &\le \frac{1}{\sqrt{2\pi}} \exp\left((N+n)\,(1+\ln(2nc) - \ln(N+n))\right).\end{aligned}$$

Let us pick N so that

$$1+\ln(2nc) - \ln(N+n) \le -1,$$

or equivalently,

$$N \ge N_0 = \max(1, n(2e^2c-1)).$$

Thus, for $N \ge N_0$, the absolute error

$$|E_N| \le \frac{1}{\sqrt{2\pi}} e^{-(N+n)}.$$

Now for a given error tolerance ϵ, we need to choose N such that

$$\frac{1}{\sqrt{2\pi}} e^{-(N+n)} \le \epsilon,$$

or equivalently,

$$N \ge N_1 = \max\left(1, \ln\left(\frac{1}{\sqrt{2\pi}\,\epsilon}\right) - n\right).$$

Thus, the absolute error term $|E_N| \le \epsilon$ when

$$N = \max(N_0, N_1) = \max\left(1, n(2e^2c-1), \ln\left(\frac{1}{\sqrt{2\pi}\,\epsilon}\right) - n\right). \tag{A3}$$

That is, for $\frac{1}{2} \le t \le \frac{3}{2}$,

$$P\left(\frac{2nc - \chi_2^2}{\chi_{2n-2}^2} \le t\right) \approx 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + e^{-\frac{nc}{t}} \sum_{m=0}^{N} \frac{1}{(m+n-1)!}\left(\frac{nc}{t}\right)^{m+n-1} (1-t)^m$$

with the error less than ϵ for the value of N determined by (A3).

Finally, the cdf of $f_{n,c} = (2nc - \chi_2^2)/\chi_{2n-2}^2$ can be expressed as

$$F_{n,c}(t) = \begin{cases} \frac{e^{-nc}}{(1-t)^{n-1}}, & t \le 0,\\ 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + e^{-\frac{nc}{t}} \sum_{m=0}^{N} \frac{(1-t)^m}{(m+n-1)!}\left(\frac{nc}{t}\right)^{m+n-1} + \text{error}, & \frac{1}{2} \le t \le \frac{3}{2},\\ 1 - F_{2n-2}\left(\frac{2nc}{t}\right) + \frac{e^{-nc}}{(1-t)^{n-1}}\left(1 - e^{-\frac{nc(1-t)}{t}} \sum_{j=0}^{n-2} \frac{1}{j!}\left(\frac{nc(1-t)}{t}\right)^j\right), & 0 < t < \frac{1}{2} \text{ or } t > \frac{3}{2}.\end{cases}$$

The following expressions may be used to evaluate the above cdf using a programing language such as R or Fortran.

1. To compute the cdf for $\frac{1}{2} \le t \le \frac{3}{2}$, use the expression

$$\sum_{m=0}^{N}(1-t)^m \exp\left(-\frac{nc}{t} - \log(\Gamma(m+n)) + (m+n-1)(\log(nc) - \log(t))\right).$$

for

$$e^{-\frac{nc}{t}}\sum_{m=0}^{N}\frac{(1-t)^m}{(m+n-1)!}\left(\frac{nc}{t}\right)^{m+n-1}$$

2. To compute the cdf for $t > \frac{3}{2}$, use the expression

$$\frac{1}{(1-t)^{n-1}}\left(e^{-nc} - e^{-\frac{nc}{t}}\sum_{j=0}^{n-2}\frac{1}{j!}\left(\frac{nc(1-t)}{t}\right)^j\right)$$
$$= \frac{1}{(1-t)^{n-1}}\left(e^{-nc} - \sum_{j=0}^{n-2}(-1)^j \exp\left(-\frac{nc}{t} - \log(\Gamma(j+1)) + j\log\left(\frac{nc(t-1)}{t}\right)\right)\right).$$

for

$$\frac{e^{-nc}}{(1-t)^{n-1}}\left(1 - e^{-\frac{nc(1-t)}{t}}\sum_{j=0}^{n-2}\frac{1}{j!}\left(\frac{nc(1-t)}{t}\right)^j\right)$$

3. To compute the cdf for $0 < t < \frac{1}{2}$, use the expression

$$\frac{1}{(1-t)^{n-1}}\left(e^{-nc} - \sum_{j=0}^{n-2}\exp\left(-\frac{nc}{t} - \log(\Gamma(j+1)) + j\log\left(\frac{nc(1-t)}{t}\right)\right)\right).$$

for

$$\frac{e^{-nc}}{(1-t)^{n-1}}\left(1 - e^{-\frac{nc(1-t)}{t}}\sum_{j=0}^{n-2}\frac{1}{j!}\left(\frac{nc(1-t)}{t}\right)^j\right).$$

Calculation of Percentiles

The percentiles of the distribution of $f_{n,c}$ can be obtained numerically using an iterative scheme. Percentiles in Tables 1 and 2 are calculated using the bisection method (root bracketing method). The two starting values required for the iterative scheme are chosen as the endpoints of an interval with the approximate percentile based on (7) as the center. R codes to calculate the percentiles are provided in a supplementary file that can be obtained from the journal's web site or from the homepage "www.ucs.louisiana.edu/∼kxk4695/" of the first author.