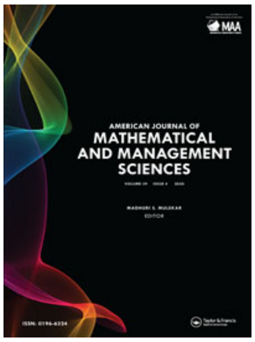

Taylor & Francis
Taylor & Francis Group

American Journal of Mathematical and Management Sciences

ISSN: 0196-6324 (Print) 2325-8454 (Online) Journal homepage: https://www.tandfonline.com/loi/umms20

Tests and Confidence Intervals for the Mean of a Zero-Inflated Poisson Distribution

Dustin Waguespack, K. Krishnamoorthy & Meesook Lee

To cite this article: Dustin Waguespack, K. Krishnamoorthy & Meesook Lee (2020) Tests and Confidence Intervals for the Mean of a Zero-Inflated Poisson Distribution, American Journal of Mathematical and Management Sciences, 39:4, 383-390, DOI: 10.1080/01966324.2020.1777914

To link to this article: https://doi.org/10.1080/01966324.2020.1777914

Published online: 22 Jun 2020.

AMERICAN JOURNAL OF MATHEMATICAL AND MANAGEMENT SCIENCES
2020, VOL. 39, NO. 4, 383–390
https://doi.org/10.1080/01966324.2020.1777914

Taylor & Francis
Taylor & Francis Group

Check for updates

Tests and Confidence Intervals for the Mean of a Zero-Inflated Poisson Distribution

Dustin Waguespack[a], K. Krishnamoorthy[a], and Meesook Lee[b]

[a]Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA, USA; [b]Department of Mathematical Sciences, McNeese State University, Lake Charles, LA, USA

ABSTRACT
The zero-inflated Poisson (ZIP) model is often postulated for count data that include excessive zeros. This ZIP distribution can be regarded as the mixture of two distributions, one that degenerate at zero and another is Poisson. Unlike the Poisson mean, the mean of the ZIP distribution is product of the mixture parameter and the Poisson parameter, and is not simple to make inference on the ZIP mean. In this article, the problem of making inference on the mean of a ZIP distribution is addressed. Confidence intervals based on the likelihood approach and bootstrap approach are provided. Signed likelihood ratio test for one-sided hypothesis is also developed. Proposed methods are evaluated for their properties by Monte Carlo simulation. Methods are illustrated using two examples.

KEYWORDS AND PHRASES
Bootstrap; coverage probability; maximum likelihood estimates; precision; Wald method

1. Introduction

The zero-inflated Poisson (ZIP) distribution is used to model data that fit a regular Poisson distribution with the exception of the presence of excess zero values. The ZIP distribution may be considered as a mixture of a distribution degenerate at zero with probability ξ and a Poisson distribution with parameter λ. The probability mass function (pmf) of the ZIP distribution is given by

$$f(x|\xi,\lambda) = \begin{cases} \xi + (1-\xi)e^{-\lambda}, & x = 0 \\ (1-\xi)\frac{e^{-\lambda}\lambda^x}{x!}, & x = 1,2,\ldots. \end{cases} \tag{1}$$

The mean of the distribution with the above pmf is given by $(1-\xi)\lambda$. The ZIP model was first described by Singh (1963) via a biological model in which the category "no offspring" combined the categories of "sterile" and "fertile with no offspring." This situation occurs commonly in biological systems where Poisson counts only appear when the system works properly, and additional zero counts are observed when the system fails (Thas & Rayner, 2005). ZIP model has been routinely used for the analysis of such data; see Leroux and Puterman (1992), Bohning et al. (1999) and Dietz and Bohning (2000).

CONTACT K. Krishnamoorthy krishna@louisiana.edu Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, USA.

Several author have addressed the problem of estimating the parameters using the likelihood approach and the large sample theory. For example, see Gupta et al. (1996) and the references therein. In general, the Wald approach with the maximum likelihood estimates and their asymptotic variances was used to find confidence intervals (CIs) for the parameters ξ and λ. Such approach depends on the model assumption that the data are ZIP distributed. However, there are situations where some tests indicate that the data are ZIP distributed while other rigorous approaches indicate that the data are not ZIP distributed. For example, Douglas et al. (1994) fitted the fetal lamb data of Leroux and Puterman (1992), which is also given in Example 2 of this paper, to several distributions and based on goodness-of-fit tests, they found that the ZIP distribution fits the data substantially better than the Poisson distribution. The score test of Van den Broeck (1995) also indicated that the fetal lamb data are not Poisson distributed and the hypothesis of ZIP distribution is tenable. Thas and Rayner (2005) developed several one-sample goodness-of-fit tests for testing ZIP against a large class of alternatives. They have applied their three new goodness-of-fit tests to fetal lamb data with a class of alternatives. Their tests clearly indicated that the data are not ZIP distributed. Thus, a method of finding a CI for the mean when the model assumption is somewhat violated is needed.

In this article, we investigate the CIs for the mean of a ZIP distribution based on the Wald approach and the signed-likelihood ratio test. As an alternative approach, we also propose CIs based on the bootstrap approach which are valid even when model assumption is violated. We describe all three methods in the following sections and compare them with respect to the coverage probabilities and expected widths. On the basis of our simulation studies, we find that the bootstrap is not only simple to implement, but also provide CIs that are narrower than the likelihood CIs. Furthermore, the bootstrap approach can be used to find CI on the basis of a frequency table that summarizes a count data set. The methods are illustrated using two real count data.

2. Likelihood Approach

Let n_i denote the observed frequency on the count i, $i = 0, 1, 2, \ldots$. Let $n = \sum n_i$ denote the sample size. Let $\bar{x} = \sum i n_i / n$ denote the sample mean. The log-likelihood function based on the data can be expressed as

$$\ln(LF) \propto n_0 \ln\left(\xi + (1-\xi)e^{-\lambda}\right) + (n - n_0)[\ln(1-\xi) - \lambda] + n\bar{x}\ln(\lambda). \tag{2}$$

The partial derivative $\partial \ln(LF)/\partial\xi = 0$ yields the equation

$$\xi + (1-\xi)e^{-\lambda} = n_0/n. \tag{3}$$

Using this equation in $\partial \ln(LF)/\partial\lambda = 0$, it can be easily checked that

$$(1-\xi)\lambda = \bar{x}. \tag{4}$$

Notice that the MLE of the mean $(1-\xi)\lambda$ is simply the sample mean. Solving (4) for ξ, and substituting the expression for ξ in (3), we arrive at the equation

$$\lambda\left(1 - \frac{n_0}{n}\right) + \bar{x}\left(e^{-\lambda} - 1\right) = 0. \tag{5}$$

The MLE $\hat{\lambda}$ of λ is the root of the above equation, which can be obtained iteratively using the Newton-Raphson scheme

$$\lambda^{(1)} = \lambda^{(0)} - \frac{\lambda^{(0)}\left(1 - \frac{n_0}{n}\right) - \bar{x}\left(1 - e^{-\lambda^{(0)}}\right)}{\left(1 - \frac{n_0}{n}\right) - \bar{x}e^{-\lambda^{(0)}}} \tag{6}$$

using $\lambda^{(0)} = \bar{x}/(1 - n_0/n)$ as the initial value. It then follows that $\hat{\xi}$, the MLE for ξ, can be computed using $\hat{\lambda}$ in (4) as $\hat{\xi} = 1 - \bar{x}/\hat{\lambda}$.

2.1. Likelihood-Based Confidence Interval for the Mean

An estimate of the variance of $\left(1 - \hat{\xi}\right)\hat{\lambda}$ can be obtained from the Fisher information matrix. Let $\hat{\nu}\left(\hat{\xi}, \hat{\lambda}\right)$ denote the estimate of the variance of $\left(1 - \hat{\xi}\right)\hat{\lambda}$ based on the Fisher information matrix; see the appendix. Then an asymptotic $100(1 - \alpha)$ confidence interval for the mean is given by

$$\bar{x} \pm z_{1-\alpha/2}\sqrt{\hat{\nu}\left(\hat{\xi}, \hat{\lambda}\right)}, \tag{7}$$

where z_p denote the pth quantile of the standard normal distribution.

2.2. Signed Likelihood Ratio Test

Let $\mu = (1 - \xi)\lambda$, and consider testing the mean

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_a : \mu > \mu_0, \tag{8}$$

where μ_0 is a specified value. To derive the likelihood ratio test (LRT) for the above hypotheses, we write the log-likelihood function under H_0 as

$$l(\lambda|\mu_0, \bar{x}) = n_0 \ln\left(1 - \mu_0/\lambda + \mu_0 e^{-\lambda}/\lambda\right) + (n - n_0)[\ln(\mu_0/\lambda) - \lambda] + n\bar{x}\ln(\lambda). \tag{9}$$

The partial derivative $\partial l(\lambda|\mu_0, \bar{x})/\partial\lambda = 0$ yields

$$f_c(\lambda) = n\left(\lambda - \mu_0 + \mu_0 e^{-\lambda}\right)(\bar{x} - \lambda - 1) + n_0\lambda(\lambda - \mu_0 + 1) = 0.$$

Using

$$f_c'(\lambda) = n\left[(\bar{x} - \lambda)\left(1 - \mu_0 e^{-\lambda}\right) + (\mu_0 - \lambda - 1)\right] + n_0(2\lambda - \mu_0 + 1),$$

the following Newton-Raphson scheme

$$\lambda^{(1)} = \lambda^{(0)} - f_c\left(\lambda^{(0)}\right)/f_c'\left(\lambda^{(0)}\right),$$

can be used to solve the equation $f_c(\lambda) = 0$. On the basis of our extensive simulation studies we find the initial value $\lambda^{(0)} = \max\{\hat{\lambda}, \mu_0/(1 - n_0/n)\}$ makes the above scheme converge faster. The root of the equation, denoted by $\hat{\lambda}_c$, is the constrained MLE of λ. The constrained MLE of ξ is given by $\hat{\xi}_c = 1 - \mu_0/\hat{\lambda}_c$.

The signed likelihood ratio test (SLRT) statistic is defined by

$$T = \operatorname{sign}(\bar{x} - \mu_0)\left(-2\left(l\left(\hat{\lambda}_c|n_0, \bar{x}\right) - l\left(\hat{\lambda}, \hat{\xi}|n_0, \bar{x}\right)\right)\right)^{1/2}, \tag{10}$$

which follows a standard normal distribution for large n.

A CI for the mean can be obtained by inverting the SLRT numerically. In particular, for a given data and the confidence level $(1 - \alpha)$, the right endpoint of the CI is determined by the value of μ_0 for which the test statistic T in (10) is equal to $-z_{1-\alpha/2}$ and the left endpoint is the value of μ_0 for which $T = z_{1-\alpha/2}$, where z_p is the 100pth percentile of the standard normal distribution. See Examples 1 and 2.

Remark. In a general setup, DiCiccio et al. (2001) have proposed a modification to the SLRT statistic so that the test based on the modified statistic is third-order accurate in the sense that the approximation to the standard normal distribution is in the order $O(n^{-3/2})$. Specifically, for our present problem the modification is to standardize the SLRT statistic T in (10) as $\left(T - \widehat{E(T)}\right)/\sqrt{\widehat{\mathrm{Var}}(T)}$, where $\widehat{E(T)}$ and $\widehat{\mathrm{Var}}(T)$ are estimates of the mean and variance of the statistic T, respectively. Specifically, $E(T)$ and $\mathrm{Var}(T)$ are estimated using the simulated samples from the ZIP$\left(\hat{\lambda}_c, \hat{\xi}_c\right)$, where $\hat{\lambda}_c$ and $\hat{\xi}_c$ are the constrained MLEs under the hypothesis that $(1 - \xi)\lambda = \mu_0$. However, in our present problem such modification has offered a little or no improvement over the SLRT, and so we have not pursued modification in our study.

3. Bootstrap Approach

The bootstrap inference is based on bootstrap samples, drawn with replacement, from the given sample from a population. The bootstrap inference does not depend on the model assumption. From the original sample of n observations, we sample with replacement a large number of samples, each of size n. Such samples are usually referred to as the bootstrap samples. Inference on a population parameter can be obtained from the collection of statistics calculated from the set of bootstrap samples. For example, appropriate percentiles of the means of the bootstrap samples can be used to interval estimate the population mean.

3.1. Bootstrap Confidence Interval

The algorithm for implementing the bootstrap method to find a confidence interval for the mean of a ZIP distribution is described below.

Bootstrap Algorithm

1. Arrange the data as shown below:

$$S = \underbrace{0, 0, \ldots, 0,}_{n_0}\ \underbrace{1, 1, \ldots, 1,}_{n_1}\ \underbrace{2, 2, \ldots, 2,}_{n_2}\ \ldots.$$

2. Select a bootstrap sample (with replacement) of size $n = \sum n_i$ from the above original sample. For example, use `sample(S, n, replace=T)` in R to select bootstrap samples of size n

Table 1. Coverage probabilities and [expected widths] of 95% likelihood CIs for the mean.

	$\lambda=1$		$\lambda=2$		$\lambda=3$		$\lambda=4$		$\lambda=5$	
ξ	L-CI	B-CI	L-CI	B-CI	L-CI	B-CI	L-CI	B-CI	L-CI	B-CI
					$n=30$					
.1	.928[.709]	.942[.687]	.944[1.04]	.940[1.02]	.944[1.35]	.940[1.30]	.924[1.58]	.941[1.56]	.919[1.80]	.942[1.81]
.2	.934[.705]	.939[.672]	.943[1.10]	.942[1.04]	.949[1.44]	.939[1.36]	.951[1.74]	.949[1.67]	.945[2.02]	.942[1.97]
.3	.933[.692]	.937[.659]	.947[1.10]	.945[1.04]	.940[1.46]	.940[1.39]	.947[1.79]	.940[1.74]	.951[2.11]	.946[2.06]
.4	.926[.667]	.934[.629]	.942[1.08]	.940[1.02]	.943[1.45]	.942[1.39]	.943[1.79]	.938[1.74]	.945[2.14]	.947[2.09]
.5	.920[.631]	.933[.587]	.938[1.04]	.943[.977]	.938[1.40]	.940[1.35]	.935[1.75]	.936[1.71]	.943[2.10]	.944[2.05]
					$n=40$					
.1	.944[.619]	.943[.600]	.941[.927]	.952[.887]	.954[1.19]	.942[1.14]	.943[1.40]	.944[1.36]	.943[1.60]	.949[1.58]
.2	.936[.616]	.947[.590]	.951[.961]	.944[.903]	.950[1.25]	.942[1.19]	.946[1.50]	.948[1.46]	.946[1.76]	.944[1.72]
.3	.937[.603]	.940[.574]	.946[.963]	.945[.905]	.948[1.27]	.949[1.21]	.946[1.55]	.945[1.50]	.943[1.83]	.934[1.79]
.4	.932[.581]	.938[.549]	.946[.946]	.945[.887]	.948[1.26]	.944[1.20]	.944[1.56]	.949[1.52]	.943[1.85]	.940[1.83]
.5	.924[.551]	.936[.517]	.947[.910]	.949[.853]	.945[1.22]	.946[1.17]	.939[1.52]	.945[1.48]	.938[1.82]	.946[1.79]
					$n=50$					
.1	.942[.554]	.941[.539]	.947[.834]	.946[.798]	.953[1.06]	.952[1.03]	.951[1.26]	.946[1.22]	.946[1.45]	.952[1.42]
.2	.942[.552]	.948[.530]	.954[.862]	.946[.815]	.953[1.12]	.943[1.07]	.947[1.35]	.945[1.31]	.945[1.57]	.945[1.54]
.3	.943[.541]	.943[.516]	.952[.864]	.946[.815]	.952[1.14]	.946[1.09]	.948[1.39]	.949[1.36]	.946[1.64]	.949[1.62]
.4	.938[.522]	.950[.495]	.949[.849]	.948[.801]	.948[1.13]	.949[1.09]	.944[1.40]	.950[1.36]	.944[1.66]	.940[1.64]
.5	.940[.495]	.941[.462]	.947[.815]	.947[.766]	.946[1.10]	.944[1.06]	.945[1.36]	.943[1.33]	.942[1.64]	.947[1.61]
					$n=70$					
.1	.948[.470]	.946[.457]	.951[.707]	.949[.681]	.953[.903]	.952[.868]	.954[1.07]	.947[1.04]	.949[1.23]	.952[1.20]
.2	.945[.468]	.945[.451]	.959[.730]	.947[.691]	.952[.948]	.948[.906]	.949[1.14]	.948[1.11]	.945[1.33]	.946[1.31]
.3	.947[.459]	.943[.423]	.950[.733]	.948[.691]	.953[.962]	.948[.925]	.950[1.18]	.944[1.15]	.948[1.39]	.947[1.36]
.4	.948[.444]	.945[.440]	.954[.720]	.949[.678]	.952[.955]	.949[.920]	.950[1.18]	.951[1.16]	.942[1.41]	.944[1.39]
.5	.944[.421]	.950[.397]	.952[.691]	.944[.656]	.951[.928]	.948[.894]	.952[1.16]	.947[1.13]	.944[1.38]	.948[1.36]

3. Compute the mean, say, $\bar{x}_*$ of the bootstrap sample.
4. Repeat steps 2 – 3 for a large number of times, say, 10,000

The lower and upper $100\alpha/2$ percentiles of 10,000 $\bar{x}_*$'s form a $100(1-\alpha)$ percent confidence interval for the mean $(1-\xi)\lambda$.

4. Simulation Study

To judge the accuracy of the proposed CIs and to compare them, we estimated the coverage probabilities and expected widths of the CIs for $\lambda = 1, 2, ..., 5, \xi = .1, .2, .3, .4, .5$ and for some sample sizes of 30 or more. For our simulation studies, samples from a ZIP distribution were generated using the R function "rzip" which is included in R package ZIM. The results for the likelihood CI (L-CI) and the bootstrap CI (B-CI) are reported in Table 1. Examination of the coverage probabilities and the expected widths clearly indicates that both CIs are very similar in terms of coverage probabilities, having minimum coverage probability of 0.93. Coverage probabilities of these CIs increase to the nominal level 0.95 with increasing λ and/or n. Comparison of these two CIs on the basis of expected width shows that the bootstrap CIs are shorter than the likelihood CIs for all the cases considered. We also notice that the difference between expected widths decreasing with increasing sample size or parameter λ. As the bootstrap CI does not depend on the model assumption, easy to compute and better than the likelihood CI in terms of coverage probability and precision, the bootstrap CI could be recommended for applications whether or not the data are ZIP distributed.

Table 2. Type I error rates and powers of the SLRT test $H_0: \mu = \mu_0$ vs. $H_a: \mu > \mu_0$; $\mu = (1-\xi)\lambda$; $\mu_0 = (1-\xi)$.

ξ	$\lambda = 1$	$\lambda = 1.3$	$\lambda = 1.6$	$\lambda = 2$
		$n = 30$		
.1	.048	.364	.799	.984
.2	.049	.328	.729	.956
.3	.048	.295	.639	.906
.4	.049	.259	.556	.833
.5	.048	.217	.461	.732
		$n = 40$		
.1	.049	.479	.890	.997
.2	.051	.416	.828	.987
.3	.048	.355	.745	.960
.4	.048	.304	.653	.912
.5	.046	.259	.560	.838
		$n = 50$		
.1	.045	.517	.943	.999
.2	.051	.481	.895	.996
.3	.046	.408	.820	.983
.4	.048	.348	.737	.955
.5	.045	.291	.637	.899

We also estimated the type I error rates and powers of the signed-likelihood ratio test for $H_0: \mu \le \mu_0$ vs. $H_a: \mu > \mu_0$. The value of μ_0 is assumed to be $1 - \xi$ and the true mean is taken as $\mu = (1-\xi)\lambda$ so that the probability of rejection at $\lambda = 1$ is the true type I error rate. The type I error rates and the powers are reported in Table 2. We observe from this table that the type I error rates are very close to the nominal level 0.05 for sample sizes of 30 or more. We also see that the power is increasing with increasing λ and/or increasing sample size. Overall, we see that the test can be safely used as long as the sample size or the total count is 30 or more and ξ is no more than 0.5.

5. Examples

Example 1. The Annual Report of the pension fund S.P.P. for 1952 (cited in Kennedy & Gentle, 1980) reported that some 4,075 widows received pensions from the fund. The following Table 3 shows the number of widows and their children who were entitled to support from the fund.

Notice that the number of widows for the count 0 is too large, and so the data are not consistent with being a sample from a Poisson distribution. Alternatively, a ZIP model was postulated assuming some widows were observed with probability ξ from a population π_0 of widows with no children, and others were observed with probability $1 - \xi$ from a population π of widows with none or some children which can be modeled by a Poisson distribution with parameter λ.

Suppose the reported data is a sample from a mixture distribution, and we like to estimate the mean number of children per widow who received the support. The MLEs based on the data are $\hat{\lambda} = 1.03784$ and $\hat{\xi} = .61506$. The 95% likelihood CI for the mean is (.3733,.4257). That is, the mean number of children per 100 widows who were entitled for support fund is between 37 and 43 with confidence 95%. The 95% CI for the mean based on 100,000 bootstrap samples is (.3750,.4247), which is similar to the

Table 3. Data on widows with children who received pension fund.

No. of children, x_i:	0	1	2	3	4	5	6
No. of widows, n_i:	3062	587	284	103	33	4	2

Table 4. Count data from fetal lamb movements.

Outcome	0	1	2	3	4	5	6	7
Count	182	41	12	2	2	0	0	1

likelihood CI. To calculate the 95% CI based on the signed-likelihood ratio test, the SLRT statistic (10) is 1.96 at $\mu_0 = .3752$, and is -1.96 at $\mu_0 = .4250$. So the 95% CI for the mean based on the SLRT is (.3752,.4250). Notice that all three CIs are in agreement.

Example 2. In this example, we shall use the fetal lamb data of Leroux and Puterman (1992), which is reproduced here in Table 4. A more formal way to choose between the Poisson and the ZIP distribution is to apply the score test. On the basis of a score test, Van den Broeck (1995) has concluded that the data are not Poisson distributed, but they are ZIP distributed. Thas and Rayner (2005) have applied their new tests with a class of alternatives, and concluded that the data not ZIP distributed. Even though our likelihood approaches depend on the model assumption, the bootstrap approach does not, and so we shall use the data to check if there is any disparity among the results of different methods.

For these data, the MLEs are $\hat{\lambda} = .84728$ and $\hat{\xi} = .57708$. The 95% likelihood CI for the mean is (.2617,.4550). The bootstrap CI for the mean, on the basis of 10,000 bootstrap samples, is (.2625,.4667). To find the 95% CI for the mean based on SLRT, we found the SLRT statistic T in (10) is −1.96 at $\mu_0 = .4605$ and it is 1.96 at $\mu_0 = .2744$ and the 95% CI for the mean is (.2744,.4605). Notice that the likelihood CIs are close to the bootstrap CI whether or not the data are ZIP distributed.

Acknowledgements

The authors are grateful to a reviewer for providing some useful comments and suggestions.

References

Bohning, D., Dietz, E., Schlattman, P., Mendonca, L., & Kirchner, U. (1999). The zero inflated Poisson model and the decayed, missing and filled teeth in dental epidemiology. *Journal of the Royal Statistical Society: Series A (Statistics in Society)*, *162*(2), 195–209. https://doi.org/10.1111/1467-985X.00130

DiCiccio, T. J., Martin, M. A., & Stern, S. E. (2001). Simple and accurate one-sided inference from signed roots of likelihood ratios. *Canadian Journal of Statistics*, *29*(1), 67–76. https://doi.org/10.2307/3316051

Dietz, E., & Bohning, D. (2000). On estimation of Poisson parameter in zero-modified Poisson models. *Computational Statistics & Data Analysis*, *34*(4), 441–459. https://doi.org/10.1016/S0167-9473(99)00111-5

Douglas, J., Leroux, B., & Puterman, M. (1994). Empirical fitting of discrete distributions. *Biometrics*, *50*(2), 576–579. https://doi.org/10.2307/2533403

Gupta, P. L., Gupta, R. C., & Tripathi, R. C. (1996). Analysis of zero-adjusted count data. *Computational Statistics & Data Analysis*, *23*(2), 207–218. https://doi.org/10.1016/S0167-9473(96)00032-1

Kennedy, W., & Gentle, J. (1980). *Statistical Computing*. Marcel Dekker.

Leroux, B., & Puterman, M. (1992). Maximum-penalized likelihood estimation for independent and Markov-dependent mixture models. *Biometrics*, *48*(2), 545–588. https://doi.org/10.2307/2532308

Singh, S. (1963). A note on inflated Poisson distribution. *Journal of the Indian Statistical Association*, *1*, 140–144.

Thas, O., & Rayner, J. C. W. (2005). Smooth tests for the zero-inflated Poisson distribution. *Biometrics*, *61*(3), 808–815. https://doi.org/10.1111/j.1541-0420.2005.00351.x

Van den Broeck, J. (1995). A score test for zero inflation in a Poisson distribution. *Biometrics*, *51*, 738–743.

Appendix

Let $P_0 = \xi + (1-\xi)e^{-\lambda}$. Using $E(n_0) = nP_0$ and $E(\bar{x}) = (1-\xi)\lambda$, the components $I_{ij}(\xi,\lambda)$ of the Fisher information matrix $\mathcal{I}(\xi,\lambda)$ can be expressed as follows:

$$\mathcal{I}_{11}(\xi,\lambda) = -E\frac{\partial^2 l(\xi,\lambda|x)}{\partial\xi^2} = E\left(\frac{n_0\left(1-e^{-\lambda}\right)^2}{P_0^2} + \frac{n-n_0}{(1-\xi)^2}\right)$$

$$\mathcal{I}_{12}(\xi,\lambda) = -E\left(\frac{\partial^2 l(\xi,\lambda|x)}{\partial\xi\partial\lambda}\right) = -E\left(\frac{n_0 e^{-\lambda}}{P_0^2}\right)$$

$$\mathcal{I}_{22}(\xi,\lambda) = -E\left(\frac{\partial^2 l(\xi,\lambda|x)}{\partial\lambda^2}\right) = E\left(\frac{n\bar{x}}{\lambda^2} - \frac{n_0\xi(1-\xi)e^{-\lambda}}{P_0^2}\right)$$

By replacing (ξ,λ) by the MLE $\left(\hat{\xi},\hat{\lambda}\right)$, we get the observed Fisher information matrix as

$$\mathcal{I}\left(\hat{\xi},\hat{\lambda}\right) = \begin{pmatrix} -\frac{n_0\left(1-e^{-\hat{\lambda}}\right)^2}{\hat{P}_0^2} + \frac{n-n_0}{\left(1-\hat{\xi}\right)^2} & \frac{n_0 e^{-\hat{\lambda}}}{\hat{P}_0^2} \\ -\frac{n_0 e^{-\hat{\lambda}}}{\hat{P}_0^2} & \frac{n\bar{x}}{\hat{\lambda}^2} - \frac{n_0\hat{\xi}\left(1-\hat{\xi}\right)e^{-\hat{\lambda}}}{\hat{P}_0^2} \end{pmatrix}.$$

An estimate of the asymptotic variance-covariance matrix is given by $\mathcal{I}^{-1}\left(\hat{\xi},\hat{\lambda}\right)$. To find an estimate of the asymptotic variance of the MLE $\left(1-\hat{\xi}\right)\hat{\lambda}$, we note that the gradient of the MLE is $\nabla\left(1-\hat{\xi}\right)\hat{\lambda} = \left(-\hat{\lambda}, 1-\hat{\xi}\right)$. Thus, we obtain an estimate of the variance of the MLE as

$$\hat{\nu}\left(\hat{\xi},\hat{\lambda}\right) = \left(-\hat{\lambda}, 1-\hat{\xi}\right)'\mathcal{I}^{-1}\left(-\hat{\lambda}, 1-\hat{\xi}\right).$$