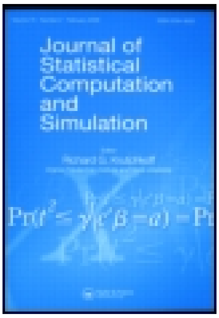

Journal of Statistical Computation and Simulation
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gscs20

New approximate confidence intervals for the difference between two Poisson means and comparison

K. Krishnamoorthy [a] & Meesook Lee [b]
[a] Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA, 70504, USA
[b] South Louisiana Community College, Lafayette, LA, 70504, USA

Available online: 22 May 2012

Journal of Statistical Computation and Simulation
iFirst, 2012, 1–12

New approximate confidence intervals for the difference between two Poisson means and comparison

K. Krishnamoorthy[a]* and Meesook Lee[b]

[a]*Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, USA;*
[b]*South Louisiana Community College, Lafayette, LA 70504, USA*

(*Received 13 January 2012; final version received 16 April 2012*)

The problem of estimating the difference between two Poisson means is considered. A new moment confidence interval (CI), and a fiducial CI for the difference between the means are proposed. The moment CI is simple to compute, and it specializes to the classical Wald CI when the sample sizes are equal. Numerical studies indicate that the moment CI offers improvement over the Wald CI when the sample sizes are different. Exact properties of the CIs based on the moment, fiducial and hybrid methods are evaluated numerically. Our numerical study indicates that the hybrid and fiducial CIs are in general comparable, and the moment CI seems to be the best when the expected total counts from both distributions are two or more. The interval estimation procedures are illustrated using two examples.

Keywords: confidence limits; exact coverage probability; Poisson rates; ratio of Poisson means

1. Introduction

Poisson distributions are routinely used to model the number of random occurrences of an event over a time interval or a specified space. Poisson model is appropriate to describe the distribution of counts of rare events, and so it is sometimes referred to as the law of rare events. If the mean rate of occurrence of an event is λ, then the probability distribution of the number of occurrences X can be modelled by a Poisson distribution with mean λ, say, Poisson(λ). In this case, the probability mass function of X is given by

$$P(X = x|\lambda) = \frac{\lambda^x \, e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots, \lambda > 0. \tag{1}$$

Let X_i be the number of random occurrences of an event from a sample of n_i units (or over a period of time t_i) with mean rate λ_i, so that $X_i \sim \text{Poisson}(n_i\lambda_i)$, $i = 1, 2$. The problem of interest here is to find confidence intervals (CIs) for the difference $\lambda_1 - \lambda_2$ based on X_1 and X_2.

The problem of estimating or testing the difference between Poisson means commonly arise in biological, epidemiological and medical sciences. For example, one may want to compare the incident rates of a disease in a treatment group and control group, where the incident rate is defined

*Corresponding author. Email: krishna@louisiana.edu

ISSN 0094-9655 print/ISSN 1563-5163 online

http://dx.doi.org/10.1080/00949655.2012.686616
http://www.tandfonline.com

as the number of events (such as death) observed divided by the time at risk during the observed period. Rothman and Greenland [1] present a specific example where two groups of women were compared to find whether those who had been examined using X-ray fluoroscopy during treatment for tuberculosis had a higher rate of breast cancer than those who had not been examined using X-ray fluoroscopy. For other-related applications and examples in health sciences, see Section 4 of this paper and the articles [2–4] that are appeared in medical journals.

There are many articles [5–8] that proposed some new tests for comparing two Poisson means and compared them with other available tests. Regarding inferential procedures for $\lambda_1 - \lambda_2$, several articles considered hypothesis testing, and provided tests, and compared it with the Wald test based on normal approximation, [6,7,9]. Even though some of the proposed tests are accurate and satisfactory (in terms of Type I error rates and powers), many of these tests are difficult to invert to obtain closed-form CIs for the difference between two Poisson means. Schwetman and Martinez [10] compared eight approximate CIs, and concluded that the Wald CI is the best. Miettinen and Nurminen [11] have developed a likelihood-based CI whose endpoints are determined by two roots of a fourth-order polynomial function. As noted by Liu *et al.* [12], it is hard to identify the two meaningless roots and so not practical to use without a numerical algorithm. Recently, Li *et al.* [13] have proposed some hybrid CIs that are based on individual CIs for λ_1 and λ_2, and made some recommendations as to the choice of CIs for applications.

In this article, we seek some simple CIs for the difference between two Poisson means that are accurate and easy to use. To this end, we propose a moment CI for the difference $\lambda_1 - \lambda_2$, which is obtained by inverting a test statistic proposed by Krishnamoorthy and Thomson [7]. This moment CI simplifies to the usual Wald CI when the sample sizes are equal, and offers appreciable improvement for some cases of unequal sample sizes. Recently, Krishnamoorthy and Lee [14] proposed a fiducial approach for finding CIs for a function of several Poisson parameters. The fiducial approach is more general, and is useful to find CIs, for example, for the ratio of two Poisson means or product of several Poisson means. So, it is of interest to see the performance of the fiducial CIs for the difference between two Poisson means. Furthermore, we also outline a 'hybrid' method of obtaining CIs proposed by Li *et al.* [13]. In Section 3, we evaluate exact coverage probabilities and expected widths of all CIs and compare them. Based on our comparison studies, some recommendations are made as to the choice of CIs for practical applications. The interval estimation procedures are illustrated using two examples in Section 4, and some concluding remarks are given in Section 5.

2. Confidence intervals

Let X_i denote the total number of Poisson events based on a sample of size n_i so that $X_i \sim \text{Poisson}(n_i\lambda_i)$, $i = 1, 2$. In the following, we shall describe some interval estimation procedures based on X_1 and X_2.

2.1. *Wald and moment CIs*

Consider testing

$$H_0 : \lambda_1 - \lambda_2 \le d \quad \text{versus } H_a : \lambda_1 - \lambda_2 > d, \tag{2}$$

where d is a specified value based on $X_1 \sim \text{Poisson}(n_1\lambda_1)$ and $X_2 \sim \text{Poisson}(n_2\lambda_2)$. Let $\hat{\lambda}_i = X_i/n_i$, $i = 1, 2$. The usual statistic for testing Equation (2) is

$$Z_w = \frac{\hat{\lambda}_1 - \hat{\lambda}_2 - d}{\sqrt{\widehat{\text{var}}(\hat{\lambda}_1 - \hat{\lambda}_2)}}, \tag{3}$$

where $\widehat{\mathrm{var}}$ is a variance estimate, which follows the standard normal distribution asymptotically. The Wald test statistic uses the variance estimate $\widehat{\mathrm{var}}(\hat{\lambda}_1 - \hat{\lambda}_2) = \hat{\lambda}_1/n_1 + \hat{\lambda}_2/n_2$. For a given level α, the Wald test rejects the H_0 in Equation (2) if $Z_w \geq z_{1-\alpha/2}$, where z_p denotes the p quantile of the standard normal distribution.

Instead of using the usual variance estimate in the test statistic, we can use 'hypothesis dependent moment estimate' as suggested by Krishnamoorthy and Thomson [7]. These authors have numerically showed that the test based on Z_w with the moment estimate performs better than the Wald test described in the preceding paragraph. To find the moment estimate for the $\mathrm{var}(\hat{\lambda}_1 - \hat{\lambda}_2) = \lambda_1/n_1 + \lambda_2/n_2$, let $w_1 = n_1/(n_1 + n_2)$ and $w_2 = 1 - w_1$. Note that, under H_0, we have $\lambda_2 = \lambda_1 - d$ and

$$E\left(\frac{X_1 + X_2}{n_1 + n_2}\right) = w_1\lambda_1 + w_2\lambda_2 = \lambda_1 - w_2 d. \tag{4}$$

Let $\hat{\lambda} = (X_1 + X_2)/(n_1 + n_2) = w_1\hat{\lambda}_1 + w_2\hat{\lambda}_2$. Then, the moment estimates of λ_1 and λ_2 based on (4) are, respectively, given by

$$\hat{\lambda}_{1m} = \hat{\lambda} + w_2 d \quad \text{and} \quad \hat{\lambda}_{2m} = \hat{\lambda} - w_1 d. \tag{5}$$

Using $\hat{\lambda}_{1m}/n_1 + \hat{\lambda}_{2m}/n_2$ as a variance estimate in (3), we propose

$$Z_m = \frac{(\hat{\lambda}_1 - \hat{\lambda}_2 - d)}{\sqrt{\hat{\lambda}_{1m}/n_1 + \hat{\lambda}_{2m}/n_2}} = \frac{(\hat{\lambda}_1 - \hat{\lambda}_2 - d)}{\sqrt{\hat{\lambda}(1/n_1 + 1/n_2) + d(1/n_1 - 1/n_2)}} \tag{6}$$

as an approximate pivotal statistic.

The estimate $\hat{\lambda}_{2m} = \hat{\lambda} - w_1 d$ could be less than or equal to zero, resulting to a negative variance estimate in Equation (6). However, we can obtain one-sided confidence limits as follows. For instance, consider testing (2), and without loss of generality assume that $d \geq 0$ (see Remark 1 below). As noted in Krishnamoorthy and Thomson [7], it can be easily verified that $\hat{\lambda}_{2m} \leq 0$ implies that $\hat{\lambda}_1 - \hat{\lambda}_2 \leq d$. In this case, no formal test is necessary and the null hypothesis in Equation (2) cannot be rejected. In other words, values of d for which $\hat{\lambda}_{2m} \leq 0$ are not in the 'rejection region'. Therefore, we shall seek the values of d for which the null hypothesis in Equation (2) is rejected. That is, assuming asymptotic normality for Z_m, we find values of d for which $Z_m > z_{1-\alpha}$, where z_q is the q quantile of the standard normal distribution. Solving this inequality $Z_m > z_{1-\alpha}$ for d, we see that for any d for which

$$L_{12} = \hat{\lambda}_1 - \hat{\lambda}_2 + \frac{z_{1-\alpha}^2}{2}\left(\frac{1}{n_1} - \frac{1}{n_2}\right) - z_{1-\alpha}\sqrt{\left(\frac{\hat{\lambda}_1}{n_1} + \frac{\hat{\lambda}_2}{n_2}\right) + \frac{z_{1-\alpha}^2}{4}\left(\frac{1}{n_1} - \frac{1}{n_2}\right)^2} > d, \tag{7}$$

the null hypothesis in Equation (2) is rejected at the level of significance α. For any $d > L_{12}$, the null hypothesis is not rejected, and so L_{12} is a $1 - \alpha$ lower confidence limit for $\lambda_1 - \lambda_2$.

Remark 1 If the specified value in Equation (2) is negative, then writing the hypotheses in Equation (2) in terms of $\lambda_2 - \lambda_1$, we can make the specified value to be positive. An upper confidence limit for $\lambda_2 - \lambda_1$ can be obtained by inverting the test for $\lambda_2 - \lambda_1$. This upper confidence limit with negative sign is the desired lower confidence limit for $\lambda_1 - \lambda_2$.

A $1 - \alpha$ upper confidence limit for $\lambda_1 - \lambda_2$ is the negative of $1 - \alpha$ lower confidence limit for $\lambda_2 - \lambda_1$, and the latter can be obtained by inverting the test for $H_0 : \lambda_2 - \lambda_1 \leq d$ versus $H_a : \lambda_2 - \lambda_1 > d$. Note that this testing problem is the same as the one for the hypotheses in

Equation (2), and the lower confidence limit for $\lambda_2 - \lambda_1$ is the one for $\lambda_1 - \lambda_2$ with subscripts (1,2) replaced by (2,1). Specifically, the $1-\alpha$ lower confidence limit for $\lambda_2 - \lambda_1$ is given by

$$L_{21} = \hat{\lambda}_2 - \hat{\lambda}_1 + \frac{z_{1-\alpha}^2}{2}\left(\frac{1}{n_2} - \frac{1}{n_1}\right) - z_{1-\alpha}\sqrt{\left(\frac{\hat{\lambda}_1}{n_1} + \frac{\hat{\lambda}_2}{n_2}\right) + \frac{z_{1-\alpha}^2}{4}\left(\frac{1}{n_1} - \frac{1}{n_2}\right)^2}.$$

As noted earlier, $-L_{21}$ is the $1-\alpha$ upper confidence limit for $\lambda_1 - \lambda_2$, which is L_{12} in Equation (7) with the term $-z_{1-\alpha}$ replaced by $+z_{1-\alpha}$. A $1-\alpha$ CI for $\lambda_1 - \lambda_2$ can be readily obtained from these one-sided limits, and is given by

$$\hat{\lambda}_1 - \hat{\lambda}_2 + \frac{z_{1-\alpha/2}^2}{2}\left(\frac{1}{n_1} - \frac{1}{n_2}\right) \pm z_{1-\alpha/2}\sqrt{\left(\frac{\hat{\lambda}_1}{n_1} + \frac{\hat{\lambda}_2}{n_2}\right) + \frac{z_{1-\alpha/2}^2}{4}\left(\frac{1}{n_1} - \frac{1}{n_2}\right)^2}. \quad (8)$$

Since the above CI is based on moment estimates, we shall refer to this CI as the moment CI. Note that this moment CI in Equation (8) simplifies to the Wald CI

$$\hat{\lambda}_1 - \hat{\lambda}_2 \pm z_{1-\alpha/2}\sqrt{\left(\frac{\hat{\lambda}_1}{n_1} + \frac{\hat{\lambda}_2}{n_2}\right)} \quad (9)$$

when $n_1 = n_2$.

2.2. *Fiducial CIs*

Following the procedure of Cox [15], Krishnamoorthy and Lee [14] developed a fiducial quantity for the mean of a Poisson distribution as follows. Let k_i be an observed value of X_i, $i = 1, 2$. Noting that $P(X_i \geq k_i | n_i\lambda_i) = P(\chi^2_{2k_i} \leq 2n_i\lambda_i)$ and $P(X_i \leq k_i | n_i\lambda_i) = P(\chi^2_{2k_i+2} > 2n_i\lambda_i)$, where χ^2_a is the chi-square random variable with degrees of freedom a, we see that there is a pair of fiducial variables for λ_i, namely, $\chi^2_{2k_i}/(2n_i)$ and $\chi^2_{2k_i+2}/(2n_i)$. As χ^2_a is stochastically increasing in a, the random variable $\chi^2_{2k_i+1}/(2n_i)$, which stochastically lies between the two fiducial variables, is an approximate fiducial quantity for λ_i. Denoting this approximate fiducial quantity by Q_{λ_i}, we can find a fiducial quantity for the difference between two means as

$$Q_{\lambda_1-\lambda_2} = Q_{\lambda_1} - Q_{\lambda_2} = \frac{1}{2n_1}\chi^2_{2k_1+1} - \frac{1}{2n_2}\chi^2_{2k_2+1}. \quad (10)$$

Note that, for a given (k_1, k_2), the distribution of $Q_{\lambda_1-\lambda_2}$ does not depend on any parameter, and so its percentiles can be evaluated numerically or estimated by Monte Carlo simulation. The interval $(Q_{\lambda_1-\lambda_2;\alpha/2}, Q_{\lambda_1-\lambda_2;1-\alpha/2})$, where $Q_{\lambda_1-\lambda_2;p}$ is the p quantile of $Q_{\lambda_1-\lambda_2}$, is a $1-\alpha$ CI for $\lambda_1 - \lambda_2$.

To evaluate the percentiles of Equation (10) numerically, we note that, for $c_1 > 0$ and $c_2 > 0$,

$$P(c_1\chi^2_{f_1} - c_2\chi^2_{f_2} \leq q) = \frac{1}{2^{f_2/2}\Gamma(f_2/2)}\int_0^\infty F\left(\frac{q + c_2 y}{c_1}; f_1\right) \mathrm{e}^{-y/2} y^{(f_2/2)-1}\,\mathrm{d}y,$$

where $F(x; f)$ denotes the cumulative distribution function of χ^2_f random variable. The pth quantile of Equation (10) is the value of q for which

$$P(c_1\chi^2_{f_1} - c_2\chi^2_{f_2} \leq q) = p, \quad (11)$$

where $c_1 = 0.5n_1$, $c_2 = 0.5n_2$, $f_1 = 2k_1 + 1$ and $f_2 = 2k_2 + 1$. The root of Equation (11) can be obtained numerically.

It is interesting to note that the above fiducial CI and the Wald CI are asymptotically the same. To prove this, recall that χ^2_m is normally distributed for large m. Furthermore, as the chi-square random variables in Equation (10) are independent, $Q_{\lambda_1-\lambda_2}$ follows a $N(\mu, \sigma^2)$ distribution asymptotically, where the mean and the variance are determined by

$$\mu = E(Q_{\lambda_1-\lambda_2}) = \frac{2k_1+1}{2n_1} - \frac{2k_2+1}{2n_2} \quad \text{and} \quad \sigma^2 = \mathrm{var}(Q_{\lambda_1-\lambda_2}) = \frac{2k_1+1}{2n_1^2} + \frac{2k_2+1}{2n_2^2}.$$

Thus, for large samples, the fiducial CI can be expressed as

$$\mu \pm z_{1-\alpha/2}\sigma = (\hat{\lambda}_1 - \hat{\lambda}_2) + \frac{1}{2}\left(\frac{1}{n_1} - \frac{1}{n_2}\right) \pm z_{1-\alpha/2}\sqrt{\frac{\hat{\lambda}_1}{n_1} + \frac{\hat{\lambda}_2}{n_2} + \frac{1}{2}\left(\frac{1}{n_1^2} + \frac{1}{n_2^2}\right)}. \tag{12}$$

It is clear that the above CI approaches the Wald CI in Equation (9) as $(n_1, n_2) \to \infty$.

2.3. *Hybrid CIs*

We shall now outline a hybrid approach for finding a CI for $\lambda_1 - \lambda_2$ as described in Li *et al.* [13]. This approach essentially uses a variance estimate that depends on individual confidence limits for λ_1 and λ_2. Let (l_i, u_i) be a $1-\alpha$ CI for λ_i based on the total count X_i, $i = 1, 2$. A $1-\alpha$ CI (L, U) for $\lambda_1 - \lambda_2$ based on the individual confidence limits is given by

$$L = \hat{\lambda}_1 - \hat{\lambda}_2 - \sqrt{(\hat{\lambda}_1 - l_1)^2 + (u_2 - \hat{\lambda}_2)^2} \tag{13}$$

and

$$U = \hat{\lambda}_1 - \hat{\lambda}_2 + \sqrt{(u_1 - \hat{\lambda}_1)^2 + (\hat{\lambda}_2 - l_2)^2}. \tag{14}$$

Li *et al.* [13] compared a few hybrid CIs and other CIs, and concluded that the hybrid CI based on the Freedman–Tukey confidence limits l_i and u_i, and the one based on the Jeffreys confidence limits l_i and u_i are comparable and they are better than others, and so we shall consider only the latter for comparison purpose.

The Jeffreys CIs for the individual parameters λ_1 and λ_2 are given by

$$(l_i, u_i) = \left(\frac{1}{2n_i}\chi^2_{2k_i+1;\alpha/2}, \frac{1}{2n_i}\chi^2_{2k_i+1;1-\alpha/2}\right), \quad i = 1, 2,$$

where (k_1, k_2) is an observed value of (X_1, X_2) and $\chi^2_{m,\alpha}$ denotes the α quantile of the chi-square distribution with m degrees of freedom. Substituting these limits in Equations (13) and (14), we get a CI for $\lambda_1 - \lambda_2$, which will be referred to as the Jeffreys hybrid CI or simply hybrid CI.

3. Accuracy studies and comparison

To judge the accuracy of the moment and other CIs, we computed their coverage probabilities as follows. Let $(L(X_1, X_2), U(X_1, X_2))$ be a $1-\alpha$ CI for $\eta = \lambda_1 - \lambda_2$. Then, for a given λ_1 and λ_2, the coverage probability of $(L(X_1, X_2), U(X_1, X_2))$ is given by

$$\sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty} \frac{\mathrm{e}^{-n_1\lambda_1}(n_1\lambda_1)^{x_1}}{x_1!}\frac{\mathrm{e}^{-n_2\lambda_2}(n_2\lambda_2)^{x_2}}{x_2!} I_{[(L(x_1,x_2),U(x_1,x_2))]}(\eta), \tag{15}$$

where $I_{[A]}(x)$ is the indicator function. For an accurate CI, the above coverage probability should be close to the nominal level $1-\alpha$. If the coverage probabilities of a CI are often larger than

$1-\alpha$, then it is classified as conservative; if they are less than $1-\alpha$, then the CI is classified as liberal or anti-conservative. Exact expected width of a CI can be calculated using Equation (15) with the indicator function replaced by the width $U(x_1,x_2)-L(x_1,x_2)$.

The infinite sums in Equation (15) can be evaluated by first computing the probabilities at the modes of the Poisson distributions $\text{int}(n_1\lambda_1)$ and $\text{int}(n_2\lambda_2)$, and then computing the other terms using forward and backward recurrence relations for Poisson probability mass function. Evaluation of each sum in Equation (15) may be terminated once the Poisson probabilities are small, say, less than 10^{-6}.

We shall first compare the Wald, moment and the approximate fiducial (Equation (12)) CIs because they are of similar form. The coverage probabilities of all three CIs are presented in Table 1 for some small sample sizes and parameter values. The reported coverage probabilities clearly indicate that the moment CI has better coverage probabilities than other two CIs. In particular, the Wald CI is too liberal as its coverage probabilities are much smaller than the nominal levels, whereas the approximate fiducial CI is too conservative having coverage probabilities much larger than the nominal levels in some cases. We next compare the Wald and the moment CIs for some moderate but very different sample sizes. For ($n_1=10$, $n_2=50$) and ($n_1=20$, $n_2=60$), we calculated the coverage probabilities of 95% CIs and reported them in Table 2 for some small values of parameters. We observe from the reported values that the coverage probabilities of the moment CI are very close to the nominal level whereas those of the Wald CI are not so close to the nominal level 0.95, and in some cases they are much smaller than the nominal level. In view of these comparison results, we shall compare only the moment, fiducial and the hybrid CIs in the sequel.

The coverage probabilities of the moment, fiducial and the hybrid CIs are plotted in Figure 1 as a function of λ_1 while λ_2 is fixed at 0.5, 1 and 2. These three plots indicate that none of the CIs is

Table 1. Coverage probabilities of CIs.

	$1-\alpha=0.90$			$1-\alpha=0.95$			$1-\alpha=0.99$		
$(\lambda_1,n_1,\lambda_2,n_2)$	1	2	3	1	2	3	1	2	3
(1,5,1,1)	0.671	0.930	0.976	0.729	0.959	0.996	0.913	0.985	1
(1,5,1,2)	0.845	0.906	0.921	0.892	0.957	0.976	0.968	0.994	0.998
(1,10,1,1)	0.628	0.933	0.986	0.635	0.957	0.998	0.680	0.985	1
(1,7,2,1)	0.847	0.928	0.889	0.862	0.966	0.942	0.865	0.990	0.993
(5,10,6,1)	0.874	0.902	0.902	0.922	0.953	0.946	0.961	0.990	0.979

Notes: 1, Wald; 2, moment; 3, approximate fiducial.

Table 2. Coverage probabilities of 95% CIs; $n_1=10$, $n_2=50$ ($n_1=20$, $n_2=60$).

		λ_1				
λ_2	Methods	0.2	0.4	0.7	1	1.2
0.2	Wald	0.859 (0.925)	0.905 (0.940)	0.931 (0.948)	0.944 (0.950)	0.933 (0.949)
	Moment	0.961 (0.954)	0.950 (0.949)	0.937 (0.940)	0.943 (0.944)	0.942 (0.943)
0.4	Wald	0.901 (0.931)	0.917 (0.939)	0.932 (0.945)	0.937 (0.948)	0.940 (0.949)
	Moment	0.952 (0.951)	0.956 (0.952)	0.952 (0.948)	0.951 (0.950)	0.948 (0.948)
0.7	Wald	0.921 (0.937)	0.931 (0.941)	0.932 (0.944)	0.938 (0.947)	0.940 (0.948)
	Moment	0.953 (0.952)	0.953 (0.952)	0.952 (0.950)	0.951 (0.951)	0.952 (0.950)
1	Wald	0.929 (0.942)	0.934 (0.943)	0.936 (0.946)	0.939 (0.947)	0.940 (0.947)
	Moment	0.951 (0.949)	0.953 (0.949)	0.952 (0.951)	0.952 (0.949)	0.951 (0.949)
1.2	Wald	0.933 (0.942)	0.935 (0.944)	0.936 (0.945)	0.938 (0.946)	0.941 (0.947)
	Moment	0.952 (0.949)	0.951 (0.950)	0.951 (0.951)	0.950 (0.950)	0.951 (0.950)

Note: The values within parentheses are coverage probabilities when $n_1=20, n_2=60$.

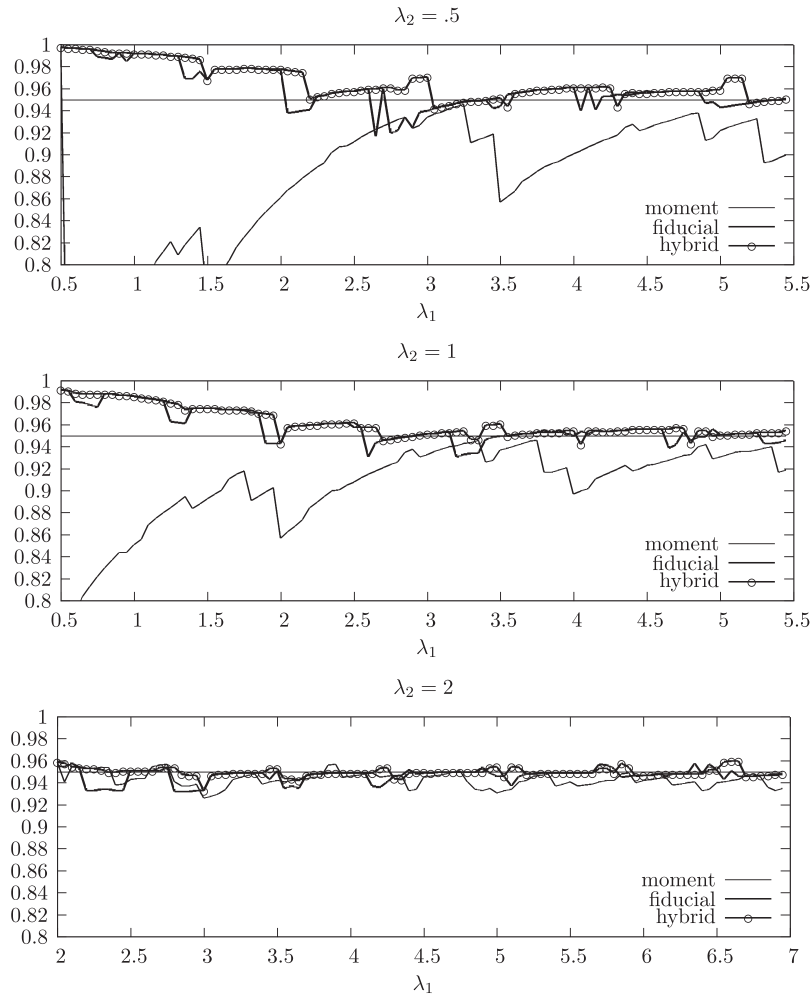

Figure 1. Coverage probabilities of 95% CIs; $n_1 = n_2 = 1$.

satisfactory for very small values of parameters. Specifically, we observe from the first two plots that, for small parameter values, the fiducial and the hybrid CIs are overly conservative, whereas the moment CI is too liberal having coverage probabilities much smaller than the nominal level. However, we see from the third plot that all three CIs perform satisfactorily controlling coverage probabilities close to the nominal level. In particular, the third plot indicates that all three CIs are satisfactory, and the hybrid and fiducial CIs perform better than the moment CI, if the mean counts of both Poisson distributions are two or more.

In Table 3, we presented coverage probabilities of 90% and 95% CIs by the moment, fiducial and hybrid methods for values of λ_1 and λ_2 ranging from 0.5 to 3.5. Here, we see that the fiducial and hybrid CIs are better than the moment CI for some small values of parameters; see the columns of $\lambda_1 = 0.75, 1, 1.25$ and the rows of $\lambda_2 = 1.5$ and 2. We once again observe that for $\lambda_1 \geq 2$ and $\lambda_2 \geq 2$, all three CIs control the coverage levels satisfactorily. In Table 4, we present coverage probabilities for $n_1 = n_2 = 4$ and for the same parameter configurations as in Table 3. For all the cases considered, note that the expected counts are two or more, and so the coverage probabilities are very close to the nominal level. However, the moment CI has an edge over other two CIs in

Table 3. Coverage probabilities of 95% and 90% (in parentheses) CIs; $n_1 = n_2 = 1$.

		λ_1								
λ_2	Methods	0.5	0.75	1	1.25	1.5	2	2.5	3	3.5
0.5	Moment	1 (0.98)	–	–	–	–	–	–	–	–
	Fiducial	1 (0.98)	–	–	–	–	–	–	–	–
	Hybrid	1 (0.98)	–	–	–	–	–	–	–	–
0.75	Moment	0.71 (0.70)	0.99 (0.96)	–	–	–	–	–	–	–
	Fiducial	0.99 (0.97)	0.99 (0.96)	–	–	–	–	–	–	–
	Hybrid	0.99 (0.97)	0.99 (0.96)	–	–	–	–	–	–	–
1	Moment	0.77 (0.73)	0.82 (0.80)	0.99 (0.94)	–	–	–	–	–	–
	Fiducial	0.99 (0.96)	0.99 (0.90)	0.99 (0.94)	–	–	–	–	–	–
	Hybrid	0.99 (0.96)	0.99 (0.95)	0.99 (0.94)	–	–	–	–	–	–
1.25	Moment	0.82 (0.71)	0.85 (0.80)	0.89 (0.86)	0.97 (0.92)	–	–	–	–	–
	Fiducial	0.99 (0.95)	0.98 (0.93)	0.98 (0.93)	0.97 (0.92)	–	–	–	–	–
	Hybrid	0.99 (0.95)	0.98 (0.93)	0.98 (0.93)	0.97 (0.92)	–	–	–	–	–
1.5	Moment	0.77 (0.76)	0.88 (0.77)	0.89 (0.83)	0.92 (0.88)	0.97 (0.91)	–	–	–	–
	Fiducial	0.98 (0.90)	0.98 (0.93)	0.97 (0.91)	0.95 (0.91)	0.97 (0.90)	–	–	–	–
	Hybrid	0.97 (0.97)	0.98 (0.93)	0.97 (0.91)	0.97 (0.91)	0.97 (0.90)	–	–	–	–
2	Moment	0.86 (0.85)	0.86 (0.84)	0.86 (0.84)	0.93 (0.84)	0.93 (0.87)	0.96 (0.90)	–	–	–
	Fiducial	0.94 (0.91)	0.97 (0.90)	0.94 (0.89)	0.96 (0.89)	0.96 (0.90)	0.96 (0.88)	–	–	–
	Hybrid	0.98 (0.91)	0.97 (0.90)	0.94 (0.94)	0.96 (0.89)	0.96 (0.90)	0.96 (0.88)	–	–	–
2.5	Moment	0.91 (0.89)	0.91 (0.89)	0.91 (0.88)	0.91 (0.87)	0.90 (0.87)	0.94 (0.88)	0.95 (0.90)	–	–
	Fiducial	0.96 (0.89)	0.95 (0.89)	0.96 (0.89)	0.95 (0.90)	0.93 (0.89)	0.95 (0.89)	0.94 (0.87)	–	–
	Hybrid	0.96 (0.89)	0.95 (0.93)	0.96 (0.89)	0.95 (0.90)	0.93 (0.92)	0.95 (0.89)	0.95 (0.87)	–	–
3	Moment	0.93 (0.87)	0.94 (0.91)	0.93 (0.89)	0.93 (0.90)	0.93 (0.89)	0.93 (0.89)	0.95 (0.89)	0.95 (0.90)	–
	Fiducial	0.94 (0.91)	0.93 (0.90)	0.95 (0.90)	0.94 (0.90)	0.94 (0.87)	0.93 (0.90)	0.95 (0.89)	0.95 (0.87)	–
	Hybrid	0.97 (0.91)	0.96 (0.88)	0.95 (0.90)	0.94 (0.91)	0.95 (0.87)	0.93 (0.91)	0.95 (0.89)	0.95 (0.87)	–
3.5	Moment	0.86 (0.84)	0.95 (0.84)	0.94 (0.87)	0.94 (0.90)	0.94 (0.89)	0.94 (0.90)	0.94 (0.90)	0.95 (0.89)	0.95 (0.90)
	Fiducial	0.94 (0.91)	0.95 (0.92)	0.95 (0.91)	0.94 (0.90)	0.95 (0.90)	0.94 (0.88)	0.95 (0.90)	0.95 (0.90)	0.95 (0.87)
	Hybrid	0.95 (0.92)	0.95 (0.92)	0.96 (0.91)	0.95 (0.87)	0.95 (0.90)	0.95 (0.87)	0.94 (0.90)	0.95 (0.90)	0.95 (0.87)

Table 4. Coverage probabilities and expected widths (in parentheses) of 95% CIs; $n_1 = 2$, $n_2 = 4$.

		λ_1						
λ_2	Methods	0.5	0.75	1	1.25	1.5	2	3
0.5	Moment	0.958 (1.88)	–	–	–	–	–	–
	Fiducial	0.951 (2.21)	–	–	–	–	–	–
	Hybrid	0.951 (2.29)	–	–	–	–	–	–
0.75	Moment	0.948 (2.13)	0.955 (2.34)	–	–	–	–	–
	Fiducial	0.950 (2.42)	0.942 (2.62)	–	–	–	–	–
	Hybrid	0.950 (2.49)	0.942 (2.68)	–	–	–	–	–
1	Moment	0.948 (2.34)	0.953 (2.54)	0.954 (2.73)	–	–	–	–
	Fiducial	0.948 (2.61)	0.945 (2.79)	0.943 (2.96)	–	–	–	–
	Hybrid	0.948 (2.68)	0.945 (2.85)	0.944 (3.02)	–	–	–	–
1.25	Moment	0.943 (2.54)	0.951 (2.73)	0.953 (2.90)	0.953 (3.06)	–	–	–
	Fiducial	0.952 (2.78)	0.941 (2.96)	0.949 (3.13)	0.947 (3.27)	–	–	–
	Hybrid	0.952 (2.85)	0.944 (3.02)	0.947 (3.17)	0.945 (3.32)	–	–	–
1.5	Moment	0.935 (2.73)	0.946 (2.90)	0.949 (3.06)	0.952 (3.21)	0.952 (3.36)	–	–
	Fiducial	0.945 (2.94)	0.947 (3.11)	0.949 (3.27)	0.946 (3.42)	0.950 (3.55)	–	–
	Hybrid	0.953 (3.01)	0.946 (3.17)	0.947 (3.32)	0.949 (3.46)	0.945 (3.60)	–	–
2	Moment	0.935 (3.06)	0.945 (3.21)	0.949 (3.36)	0.951 (3.50)	0.946 (3.63)	0.952 (3.89)	–
	Fiducial	0.952 (3.25)	0.948 (3.39)	0.950 (3.55)	0.945 (3.68)	0.947 (3.81)	0.947 (4.06)	–
	Hybrid	0.953 (3.31)	0.948 (3.46)	0.950 (3.60)	0.944 (3.73)	0.948 (3.86)	0.947 (4.10)	–
3	Moment	0.934 (3.63)	0.938 (3.76)	0.933 (3.89)	0.933 (4.01)	0.931 (4.13)	0.929 (4.35)	0.948 (4.78)
	Fiducial	0.941 (3.78)	0.951 (3.91)	0.952 (4.04)	0.947 (4.16)	0.955 (4.28)	0.933 (4.51)	0.947 (4.91)
	Hybrid	0.971 (3.84)	0.956 (3.97)	0.952 (4.09)	0.947 (4.21)	0.955 (4.32)	0.933 (4.54)	0.947 (4.95)

Table 5. Coverage probabilities and expected widths (in parentheses) of 95% CIs; $n_1 = 30$, $n_2 = 15$.

		λ_1						
λ_2	Methods	0.5	0.75	1	1.25	1.5	2	3
0.5	Moment	0.953 (0.88)	0.947 (1.01)	0.948 (1.13)	0.949 (1.24)	0.947 (1.34)	0.947 (1.52)	0.946 (1.82)
	Fiducial	0.950 (0.90)	0.949 (1.03)	0.950 (1.15)	0.950 (1.25)	0.951 (1.35)	0.951 (1.53)	0.951 (1.83)
	Hybrid	0.950 (0.91)	0.948 (1.04)	0.950 (1.15)	0.948 (1.26)	0.951 (1.35)	0.948 (1.53)	0.949 (1.83)
0.75	Moment	0.952 (0.95)	0.952 (1.08)	0.949 (1.19)	0.951 (1.29)	0.949 (1.39)	0.949 (1.56)	0.949 (1.86)
	Fiducial	0.947 (0.97)	0.946 (1.09)	0.948 (1.20)	0.950 (1.30)	0.949 (1.40)	0.950 (1.57)	0.948 (1.86)
	Hybrid	0.951 (0.98)	0.949 (1.10)	0.947 (1.21)	0.950 (1.31)	0.949 (1.40)	0.951 (1.57)	0.950 (1.87)
1	Moment	0.950 (1.01)	0.951 (1.13)	0.950 (1.24)	0.951 (1.34)	0.950 (1.43)	0.949 (1.60)	0.949 (1.89)
	Fiducial	0.948 (1.04)	0.950 (1.15)	0.949 (1.26)	0.950 (1.35)	0.948 (1.44)	0.948 (1.61)	0.949 (1.90)
	Hybrid	0.950 (1.04)	0.949 (1.16)	0.950 (1.26)	0.950 (1.36)	0.950 (1.45)	0.950 (1.61)	0.949 (1.90)
1.25	Moment	0.949 (1.08)	0.951 (1.19)	0.951 (1.29)	0.950 (1.39)	0.951 (1.48)	0.950 (1.64)	0.949 (1.93)
	Fiducial	0.948 (1.09)	0.948 (1.21)	0.948 (1.31)	0.950 (1.40)	0.950 (1.49)	0.949 (1.65)	0.949 (1.93)
	Hybrid	0.950 (1.10)	0.949 (1.21)	0.950 (1.31)	0.949 (1.40)	0.950 (1.49)	0.950 (1.65)	0.949 (1.94)
1.5	Moment	0.950 (1.13)	0.951 (1.24)	0.950 (1.34)	0.950 (1.43)	0.951 (1.52)	0.950 (1.68)	0.950 (1.96)
	Fiducial	0.950 (1.15)	0.950 (1.26)	0.950 (1.36)	0.948 (1.45)	0.949 (1.53)	0.949 (1.69)	0.950 (1.97)
	Hybrid	0.948 (1.16)	0.950 (1.26)	0.949 (1.36)	0.949 (1.45)	0.950 (1.54)	0.950 (1.69)	0.950 (1.97)
2	Moment	0.950 (1.24)	0.950 (1.34)	0.951 (1.43)	0.951 (1.52)	0.950 (1.60)	0.951 (1.75)	0.950 (2.02)
	Fiducial	0.950 (1.26)	0.948 (1.36)	0.951 (1.45)	0.949 (1.53)	0.951 (1.62)	0.948 (1.76)	0.950 (2.03)
	Hybrid	0.950 (1.27)	0.950 (1.36)	0.949 (1.45)	0.950 (1.54)	0.950 (1.62)	0.950 (1.77)	0.950 (2.04)
3	Moment	0.949 (1.43)	0.951 (1.52)	0.950 (1.60)	0.950 (1.68)	0.950 (1.75)	0.950 (1.89)	0.950 (2.15)
	Fiducial	0.948 (1.45)	0.950 (1.53)	0.949 (1.62)	0.949 (1.69)	0.950 (1.77)	0.949 (1.91)	0.951 (2.16)
	Hybrid	0.949 (1.45)	0.950 (1.54)	0.950 (1.62)	0.950 (1.70)	0.950 (1.77)	0.950 (1.91)	0.949 (2.16)

Table 6. 95% CIs for the difference between incidence rates in treatment groups losartan and captopril.

	Losartan		Captopril					
Methods	k_1	t_1	k_2	t_2	Wald	Moment	Hybrid	Fiducial[a]
Death	11	309.5	25	295.3	(−0.0887, −0.0101)	(−0.0910, −0.1053)	(−0.0885, −0.0098)	(−0.0905, −0.0107)
Pain, chest	6	309.2	5	295.9	(−0.0193, 0.0237)	(−0.0206, 0.0256)	(−0.0190, 0.0239)	(−0.0202, 0.0254)
Heart failure	22	303.7	22	288.6	(−0.0482, 0.0399)	(−0.0490, 0.0408)	(−0.0478, 0.0403)	(−0.0494, 0.0403)
Myocardial infarction	5	308.7	12	294.1	(−0.0520, 0.0022)	(−0.0546, 0.0022)	(−0.0517, 0.0025)	(−0.0545, 0.0019)

[a]Based on 100,000 simulation runs.
Notes: k_1, k_2, number of events; t_1, t_2, patient-years.

terms of precision (shorter expected widths). The moment CI has shorter expected width than other two CIs for all the cases considered.

Finally, to judge the performance of the CIs for moderate sample sizes, we evaluated the coverage probabilities and expected widths for sample sizes $n_1 = 10$ and $n_2 = 50$, and presented then in Table 5. As anticipated, all CIs have excellent coverage properties maintaining coverage probabilities very close to the nominal level. We also notice that the moment CI has shortest expected widths among all three CIs, even though the differences in expected widths are not appreciable.

Overall, our evaluation studies indicate that the fiducial and hybrid CIs are preferable for very small expected counts, and the moment CI is preferable to other two CIs when the expected counts from both distributions are two or more.

4. Examples

Example 1 To illustrate the methods in the preceding sections, we shall use the serious adverse experience data analysed in Liu *et al.* [12]. The data were collected during the course of a 48-week multi-center clinical trial to compare the tolerability of two drugs losartan and captopril. A total of 722 elderly with heart failure were randomly assigned to double-blind losartan ($n = 352$) or to captopril ($n = 370$). Various adverse events and the patient-years are reported in the study, and for illustration purpose, we present here a few events, namely, deaths, heart failure, chest pain and myocardial infarction along with patient-years in Table 6.

Note that for each type of event, the numbers of occurrences are not too small and the number of patient-years is quite large. As a result, all methods produced CIs that are very similar; see Table 6. In particular, we notice that the Wald CIs and the moment CIs are very close, and the Jeffreys hybrid CIs and the fiducial CIs are in good agreement.

Example 2 This example is taken from Jaech [16], and it involves comparison of two types of material. Three reactor fuel element failures were observed out of 310 process tubes for a given type material. In a second type material, seven failures were observed out of 3500 process tubes. Here, binomial models are more appropriate to compare the failure rates. As the Poisson distribution is a limiting form of a binomial distribution, we can apply the interval estimation procedures given earlier. Let λ_1 be the failure rate of the first-type material, and λ_2 be the failure rate of the second-type material. CIs for the difference of failure rates are given in Table 7.

Here, we observe that the computed moment, fiducial and Jeffreys hybrid intervals for the difference indicate that λ_1 is greater than λ_2, whereas the Wald interval for the difference (one-sided as well as two-sided) indicates no significant difference between λ_1 and λ_2. This example indicates that even for very large samples, the Wald CI could be inaccurate.

Table 7. 95% CIs for the difference between failure rates of two types of fuel elements.

Method	Two-sided	One-sided lower	One-sided upper
Wald	(−0.0034, 0.0187)	−0.00160	0.01695
Moment	(0.0009, 0.02573)	0.00156	0.02175
Fiducial[a]	(0.0004, 0.02364)	0.00124	0.02068
Fiducial[b]	(0.0004, 0.02375)	0.00123	0.02061
Jeffreys hybrid	(0.0005, 0.02386)	0.00130	0.02072

[a]Based on 100,000 simulation runs.
[b]Based on the roots of Equation (11).

5. Concluding remarks

We proposed a moment CI and a fiducial CI for estimating the difference between two Poisson means. The moment CI is similar to the classical Wald CI, and it coincides with the Wald CI when the sample sizes are equal. Our numerical study indicates that the Wald CI can be safely used in situations where $n_1 = n_2$ and the expected counts from both Poisson distributions are at least two. Furthermore, our studies and Example 2 indicate that all CIs that are considered in the preceding sections could perform asymptotically similar, but they do differ appreciably even for large samples. In particular, as noted in Example 2, the Wald CI could be different from others even for very large samples with unequal sample sizes. The moment CI for the difference between two Poisson means is simple to compute and offers improvement over other CIs provided expected number of total counts from each population is at least two, which is a reasonable assumption in many practical situations. Between the Jeffreys hybrid and fiducial CIs, the latter one offers slight improvement (having shorter expected widths) over the former, but computation of the fiducial CIs involves Monte Carlo simulation or numerical method. Finally, we note that both fiducial and hybrid CIs could be overly conservative (yielding CIs that are unnecessarily wide) for small values of $(n_1\lambda_1, n_2\lambda_2)$, and thus they are also inaccurate in a sense.

References

[1] K.J. Rothman and S. Greenland, *Modern Epidemiology*, Lippincott-Raven, Philadelphia, 1998.
[2] E. Agustin, R.W. Hass, and D.R. Inmaculada, *Paradoxical effect of body mass index on survival in rheumatoid arthritis role of comorbidity and systemic inflammation*, Arch. Intern. Med. 165 (2005), pp. 1624–1629.
[3] N.M. Lindor, K. Rabe, G.M. Petersen, R. Haile, G. Casey, J. Baron, S. Gallinger, B. Bapat, M. Aronson, J. Hopper, J. Jass, L. LeMarchand, J. Grove, J. Potter, P. Newcomb, J.P. Terdiman, P. Conrad, G. Moslein, R. Goldberg, A. Ziogas, H. Anton-Culver, M. de Andrade, K. Siegmund, S.N. Thibodeau, L.A. Boardman, and D. Seminara, *Lower cancer incidence in Amsterdam-I criteria families without mismatch repair deficiency*, J. Am. Med. Assoc. 293 (2005), pp. 1979–1985.
[4] P.J. Caraballo, J.A. Heit, E.J. Atkinson, M.D. Silverstein, W.M. O'Fallon, M.R. Castro, and L.J. Melton, *Long-term use of anticoagulants and the risk of fracture*, Arch. Intern. Med. 159 (1999), pp. 1750–1756.
[5] M. Gail, *Power computations for designing comparative Poisson trials*, Biometrics 30 (1974), pp. 231–237.
[6] H. Sahai and S.C. Misra, *Comparing means of two Poisson distributions*, Math. Sci. 17 (1992), pp. 60–67.
[7] K. Krishnamoorthy and J. Thomson, *A more powerful test for comparing two Poisson means*, J. Stat. Plan. Infer. 119 (2004), pp. 23–35.
[8] H.K.T. Ng, K. Gu, and M.L. Tang, *A comparative study of tests for the difference of two Poisson means*, Comput. Stat. Data Anal. 51 (2007), pp. 3085–3099.
[9] W. Shiue and L.J. Bain, *Experiment size and power comparisons for two-sample Poisson tests*, Appl. Stat. 31 (1982), pp. 130–134.
[10] N.C. Schwertman and R. Martinez, *Approximate confidence intervals for the difference in two Poisson parameters*, J. Stat. Comput. Simul. 50 (1994), pp. 235–247.
[11] O. Miettinen and M. Nurminen, *Comparative analysis of two rates*, Stat. Med. 4 (1985), pp. 213–226.
[12] G.F. Liu, J. Wang, K. Liu, and D.B. Snavely, *Confidence intervals for an exposure adjusted incident rate difference with applications to clinical trials*, Stat. Med. 25 (2006), pp. 1275–1286.
[13] H.-Q. Li, M.-L. Tang, W.-Y. Poon, and N.-S. Tang, *Confidence intervals for difference between two Poisson rates*, Comm. Stat. Simul. Comput. 40 (2011), pp. 1478–1491.
[14] K. Krishnamoorthy and M. Lee, *Inference for functions of parameters in discrete distributions based on fiducial approach: Binomial and Poisson cases*, J. Stat. Plan. Infer. 140 (2010), pp. 1182–1192.
[15] D.R. Cox, *Some simple approximate tests for Poisson variates*, Biometrika 40 (1953), pp. 354–360.
[16] J.L. Jaech, *Comparing two methods of obtaining a confidence interval for the ratio of Poisson parameters*, Technometrics 12 (1970), pp. 383–387.