Simple stably projectionless $C^*$-algebras

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Joint work with Guihua Gong
Lafayette, Louisiana, Oct, 2017
Most results of this talk is taken from a joint work with Guihua Gong and a joint work with Elliott, Gong and Niu.
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Recall that

Theorem (GLN, EGLN, TWW, more)

Let $A$ and $B$ be two unital separable simple $C^*$-algebras with finite nuclear dimension which satisfy the UCT. Then $A \cong = B$ if and only if $\text{Ell}(A) \cong = \text{Ell}(B)$.
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Theorem

Let $A$ and $B$ be two stably projectionless simple amenable $C^*$-algebras with UCT such that $\text{gTR}(A) \leq 1$, $\text{gTR}(B) \leq 1$.

Suppose that $\ker \rho_A = K_0(A)$ and $\ker \rho_B = K_0(B)$.

Then $A \sim B$ if and only if $\text{Ell}(A) \sim \text{Ell}(B)$.

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**Theorem**

Let $A$ and $B$ be two stably projectionless simple amenable $C^*$-algebras with UCT such that $g_{TR}(A) \leq 1$, $g_{TR}(B) \leq 1$. Suppose that $\ker \rho_A = K_0(A)$ and $\ker \rho_B = K_0(B)$. Then $A \cong B$ if and only if $\Ell(A) \cong \Ell(B)$. 
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Corollary

Let $A$ and $B$ be two separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT. Then $A \otimes \mathcal{Z}_0 \cong B \otimes \mathcal{Z}_0$ if and only if $\text{Ell}(A \otimes \mathcal{Z}_0) \cong \text{Ell}(B \otimes \mathcal{Z}_0)$.

Corollary

In particular, $\mathcal{Z}_0 \otimes \mathcal{Z}_0 \cong \mathcal{Z}_0$. 

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**Corollary**

Let $A$ and $B$ be two separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT.

$A \otimes \mathcal{Z}_0 \sim = B \otimes \mathcal{Z}_0$ if and only if $\text{Ell}(A \otimes \mathcal{Z}_0) \sim = \text{Ell}(B \otimes \mathcal{Z}_0)$.

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In particular, $Z_0 \otimes Z_0 \cong Z_0$. 
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Note $\ker \rho_A \subset K_0(A)$ is a subgroup of $K_0(A)$. 

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Let $\tilde{T}(A)$ be the cone of densely defined, positive lower semi-continuous traces on $A$ equipped with the topology of point-wise convergence on elements of the Pedersen ideal $P(A)$ of $A$.

Let $a \in A^+$. Suppose that $\tilde{T}(A) \neq \emptyset$.

Recall that $d\tau(a) = \lim_{\epsilon \to 0} \tau(f_\epsilon(a))$ with possible infinite value.

Note that $f_\epsilon(a) \in P(A)^+$. Therefore $\tau \mapsto d\tau(a)$ is a lower semi-continuous affine function on $\tilde{T}(A)$ (to $[0, \infty]$).

Suppose that $A$ is non-unital.

Let $a \in A^+$ be a strictly positive element.

Define $\Sigma_A(\tau) = d\tau(a)$ for all $\tau \in \tilde{T}(A)$.

$\Sigma_A$ is independent of the choice of $a$.

The lower semi-continuous affine function $\Sigma_A$ is called the scale function of $A$.
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**Theorem (EGLN)**

Let \( A \) and \( B \) be two separable simple \( C^* \)-algebras with finite nuclear dimension such that \( KK(A, A) = KK(B, B) = 0 \). Suppose that \( T(A), T(B) \neq \emptyset \). Then \( A \sim B \) if and only if \( \text{Ell}(A) \sim \text{Ell}(B) \).
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**Theorem**

Let \( B \) be a separable amenable simple \( C^* \)-algebra.
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**Theorem (EGLN)**

Let \( A \) and \( B \) be two separable simple C*-algebras with finite nuclear dimension such that \( KK(A, A) = KK(B, B) = 0 \). Suppose that \( T(A), T(B) \neq \emptyset \). Then \( A \cong B \) if and only if

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**Theorem**

Let \( B \) be a separable amenable simple C*-algebra. Then there exists \( A \) such that \( gTR(A) \leq 1 \) satisfying the UCT such that
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**Theorem**

Let $B$ be a separable amenable simple $C^*$-algebra. Then there exists $A$ such that $gTR(A) \leq 1$ satisfying the UCT such that

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Following L. Robert's term, let $C_0$ be the family of non-unital non-commutative 1-dimensional CW complices $C$ (non-unital version of Elliott-Thomsen building blocks) with $K_0(C)_+ = \{0\}$,
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for all $a \in F_2$. 

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$$

for all $a \in F_2$. Let

$$
A = \{(f, a) : f \in C([0, 1], F_2) \oplus F_1 : f(0) = \psi_0(a) \text{ and } f(1) = \psi_1(a)\}
$$

Then $K_0(A) = \{0\} = K_1(A)$ and $0 \notin \overline{T(A)}^w$ (Razak).
Let $F_1 = \mathbb{C} \oplus \mathbb{C}$, $F_2 = M_{2n}(\mathbb{C})$. For $(a, b) \in \mathbb{C} \oplus \mathbb{C} = F_1$, define

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Then $A = \{(f, x) \in \mathbb{C}([0, 1], F_2) \oplus F_1 : f(0) = \psi_0(x) \text{ and } f(1) = \psi_1(x)\}$ has the property that $K_0(A) = \{(k, -k) \in \mathbb{Z} \oplus \mathbb{Z} \}$, which is isomorphic to $\mathbb{Z}$ but $K_0(A) + = \{0\}$. Also $K_1(A) = \{0\}$. Thus $A \in \mathbb{C}_0$ but $A/ \in \mathbb{C}_0$. Let $\mathbb{C}'_0$ denote the class of all full hereditary $\mathbb{C}^*$-subalgebras of $\mathbb{C}^*$-algebras in $\mathbb{C}_0$.
Let $F_1 = \mathbb{C} \oplus \mathbb{C}$, $F_2 = M_{2n}(\mathbb{C})$. For $(a, b) \in \mathbb{C} \oplus \mathbb{C} = F_1$, define

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\[(e0.1)\]

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\]

\[(e0.2)\]
Let \( F_1 = \mathbb{C} \oplus \mathbb{C}, \quad F_2 = M_{2n}(\mathbb{C}) \). For \((a, b) \in \mathbb{C} \oplus \mathbb{C} = F_1\), define

\[
\psi_0(a, b) = \text{diag}(a, a \ldots a, b, b \ldots b, 0, 0)
\]

\[
\text{and} \quad \psi_1(a, b) = \text{diag}(a, a \ldots a, b, b \ldots b).
\]

Then

\[
A = \{(f, x) \in C([0, 1], F_2) \oplus F_1 : f(0) = \psi_0(x) \text{ and } f(1) = \psi_1(x)\}
\]

has the property that \( K_0(A) = \{(k, -k) \in \mathbb{Z} \oplus \mathbb{Z}\} \)
Let $F_1 = \mathbb{C} \oplus \mathbb{C}$, $F_2 = M_{2n}(\mathbb{C})$. For $(a, b) \in \mathbb{C} \oplus \mathbb{C} = F_1$, define

$$\psi_0(a, b) = \text{diag}(a, a...a, b, b...b, 0, 0)$$

and

$$\psi_1(a, b) = \text{diag}(a, a...a, b, b...b).$$

Then

$$A = \{(f, x) \in C([0, 1], F_2) \oplus F_1 : f(0) = \psi_0(x) \text{ and } f(1) = \psi_1(x)\}$$

has the property that $K_0(A) = \{(k, -k) \in \mathbb{Z} \oplus \mathbb{Z}\}$ which is isomorphic to $\mathbb{Z}$.
Let \( F_1 = \mathbb{C} \oplus \mathbb{C}, F_2 = M_{2n}(\mathbb{C}) \). For \((a, b) \in \mathbb{C} \oplus \mathbb{C} = F_1\), define

\[
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\psi_1(a, b) = \text{diag}(a, a...a, b, b...b).
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Then

\[
A = \{(f, x) \in C([0, 1], F_2) \oplus F_1 : f(0) = \psi_0(x) \text{ and } f(1) = \psi_1(x)\}
\]

has the property that \( K_0(A) = \{(k, -k) \in \mathbb{Z} \oplus \mathbb{Z}\} \) which is isomorphic to \( \mathbb{Z} \) but \( K_0(A)_+ = \{0\} \). Also \( K_1(A) = \{0\} \).
Let $F_1 = \mathbb{C} \oplus \mathbb{C}$, $F_2 = M_{2n}(\mathbb{C})$. For $(a, b) \in \mathbb{C} \oplus \mathbb{C} = F_1$, define

$$
\psi_0(a, b) = \text{diag}(a, a...a, b, b...b, 0, 0) \quad (e0.1)
$$

and

$$
\psi_1(a, b) = \text{diag}(a, a...a, b, b...b) \quad (e0.2)
$$

Then

$$
A = \{(f, x) \in C([0, 1], F_2) \oplus F_1 : f(0) = \psi_0(x) \text{ and } f(1) = \psi_1(x)\}
$$

has the property that $K_0(A) = \{(k, -k) \in \mathbb{Z} \oplus \mathbb{Z}\}$ which is isomorphic to $\mathbb{Z}$ but $K_0(A)_+ = \{0\}$. Also $K_1(A) = \{0\}$. Thus $A \in C_0$ but $A \notin C_0^0$. 

Let $C'$ denote the class of all full hereditary $C^*$-subalgebras of $C^*$-algebras in $C_0$, let $C_0'$ denote the class of all full hereditary $C^*$-subalgebras of $C^*$-algebras in $C_0$. 

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Simple stably projectionless $C^*$-algebras
Joint work with Guihua Gong
Lafayette, Louisiana, Oct, 2017
Let $F_1 = \mathbb{C} \oplus \mathbb{C}$, $F_2 = M_{2n}(\mathbb{C})$. For $(a, b) \in \mathbb{C} \oplus \mathbb{C} = F_1$, define

$$\psi_0(a, b) = \text{diag}(a, a...a, b, b...b, 0, 0)$$

and

$$\psi_1(a, b) = \text{diag}(a, a...a, b, b...b).$$

Then

$$A = \{(f, x) \in C([0, 1], F_2) \oplus F_1 : f(0) = \psi_0(x) \text{ and } f(1) = \psi_1(x)\}$$

has the property that $K_0(A) = \{(k, -k) \in \mathbb{Z} \oplus \mathbb{Z}\}$ which is isomorphic to $\mathbb{Z}$ but $K_0(A)_+ = \{0\}$. Also $K_1(A) = \{0\}$. Thus $A \in C_0$ but $A \notin C_0'$. Let $C_0'$ denote the class of all full hereditary $C^*$-subalgebras of $C^*$-algebras in $C_0$ let $C_0^{0'}$ denote the class of all full hereditary $C^*$-subalgebras of $C^*$-algebras in $C_0$. 

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Simple stably projectionless $C^*$-algebras

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Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$)
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$,
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$, Suppose that, for any $\epsilon > 0$,
Definition

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Definition

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Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$. Suppose that, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $b \in A_+ \setminus \{0\}$, there are $\mathcal{F}$-$\epsilon$-multiplicative c.p.c. linear maps $\phi : A \to A$ and $\psi : A \to D$ for some $C^*$-subalgebra $D \subset A$.
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$, Suppose that, for any $\epsilon > 0$, any finite subset $F \subset A$ and any $b \in A_+ \setminus \{0\}$, there are $F$-$\epsilon$-multiplicative c.p.c. linear maps $\phi : A \to A$ and $\psi : A \to D$ for some $C^*$-subalgebra $D \subset A$ such that $D \in C'_0$ (or in $C'_0$)

$$\|x - \text{diag}(\phi(x), \psi(x))\| < \epsilon \text{ for all } x \in F \cup \{a\}, \quad (e\ 0.3)$$
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$, Suppose that, for any $\epsilon > 0$, any finite subset $F \subset A$ and any $b \in A_+ \setminus \{0\}$, there are $F$-$\epsilon$-multiplicative c.p.c. linear maps $\phi : A \to A$ and $\psi : A \to D$ for some $C^*$-subalgebra $D \subset A$ such that $D \in C_0'$ (or in $C_0^{0'}$)

$$\|x - \text{diag}(\phi(x), \psi(x))\| < \epsilon \text{ for all } x \in F \cup \{a\}, \quad (e0.3)$$

$$\phi(a) \lesssim b, \quad (e0.4)$$
Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$, Suppose that, for any $\epsilon > 0$, any finite subset $F \subset A$ and any $b \in A_+ \setminus \{0\}$, there are $F$-$\epsilon$-multiplicative c.p.c. linear maps $\phi : A \to A$ and $\psi : A \to D$ for some $C^*$-subalgebra $D \subset A$ such that $D \in C'_0$ (or in $C'_0$)

\[
\|x - \text{diag}(\phi(x), \psi(x))\| < \epsilon \text{ for all } x \in F \cup \{a\}, \quad (e\,0.3)
\]

\[
\phi(a) \precsim b, \quad (e\,0.4)
\]

\[
t(f_{1/4}(\psi(a))) \geq 3/4 \text{ for all } t \in T(D).
\]
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$. Suppose that, for any $\epsilon > 0$, any finite subset $F \subset A$ and any $b \in A_+ \setminus \{0\}$, there are $F$-$\epsilon$-multiplicative c.p.c. linear maps $\phi: A \to A$ and $\psi: A \to D$ for some $C^*$-subalgebra $D \subset A$ such that $D \in C_0'$ (or in $C_0''$)

$$\|x - \text{diag}(\phi(x), \psi(x))\| < \epsilon \quad \text{for all } x \in F \cup \{a\}, \quad (e \, 0.3)$$

$$\phi(a) \preceq b, \quad (e \, 0.4)$$

$$t(f_{1/4}(\psi(a))) \geq 3/4 \quad \text{for all } t \in T(D). \quad (e \, 0.5)$$

Then we say $A \in D$ (or $D_0$).
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$, Suppose that, for any $\epsilon > 0$, any finite subset $F \subset A$ and any $b \in A_+ \setminus \{0\}$, there are $F$-$\epsilon$-multiplicative c.p.c. linear maps $\phi : A \to A$ and $\psi : A \to D$ for some $C^*$-subalgebra $D \subset A$ such that $D \in C'_0$ (or in $C'_0$) such that

\[
\|x - \text{diag}(\phi(x), \psi(x))\| < \epsilon \quad \text{for all} \quad x \in F \cup \{a\}, \tag{e 0.3}
\]
\[
\phi(a) \preceq b, \tag{e 0.4}
\]
\[
t(f_{1/4}(\psi(a))) \geq 3/4 \quad \text{for all} \quad t \in T(D). \tag{e 0.5}
\]

Then we say $A \in D$ (or $D_0$).

$W, Z_0 \in D_0$. 
Definition

(in a convenient setting) We say a non-unital separable simple $C^*$-algebra $A$ has generalized tracial rank at most one, if there exists $a \in P(A) + \{0\}$ such that $eAe \in D$, or $e$ is a non-zero projection and $g_{TR}(eAe) \leq 1$. 
Definition

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$$e_A e \in D,$$

or $e$ is a non-zero projection and $g_{TR}(e_A e) \leq 1$. 

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Simple stably projectionless $C^*$-algebras

Joint work with Guihua Gong Lafayette, Louis
Definition

(in a convenient setting) We say a non-unital separable simple $C^*$-algebra $A$ has generalized tracial rank at most one, if there exists $a \in P(A)_+ \setminus \{0\}$ such that $eAe \in \mathcal{D}$,
Definition

(in a convenient setting) We say a non-unital separable simple $C^*$-algebra $A$ has generalized tracial rank at most one, if there exists $a \in P(A)_+ \setminus \{0\}$ such that $eAe \in D$, or $e$ is a non-zero projection and $gTR(eAe) \leq 1$. 
Let \( A \in \mathcal{D} \).

(i) Then \( QT(A) = T(A) \);

(ii) \( A \) has the tracially approximate divisible property, if \( A \in \mathcal{D}_0 \);

(iii) \( A \) has strict comparison for positive elements (in usual sense).

(i.e., if \( a, b \in A^+ \) and \( d_\tau(a) < d_\tau(b) \) for all \( \tau \in T(A) \), then \( a \precsim b \).

Theorem (EGLN)

Let \( A \) be a non-unital separable simple \( C^* \)-algebra which is in \( \mathcal{D} \). Then \( A \) has stable rank one.
Theorem

Let $A \in \mathcal{D}$.

(1) Then $QT(A) = T(A)$.

(2) $A$ has the tracially approximate divisible property, if $A \in \mathcal{D}_0$.

(3) $A$ has strict comparison for positive elements (in usual sense).

(i.e., if $a, b \in A_+ +$ and $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$, then $a \lessapprox b$.)
Theorem

Let $A \in \mathcal{D}$.

1. Then $QT(A) = T(A)$;
2. $A$ has the tracially approximate divisible property, if $A \in \mathcal{D}_0$,
Theorem

Let $A \in \mathcal{D}$.

(1) Then $QT(A) = T(A)$;

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(i.e., if $a, b \in A_+$ and

\[ d_\tau(a) < d_\tau(b) \text{ for all } \tau \in \overline{T(A)}^w, \]

then $a \precsim b$.)
Theorem

Let $A \in \mathcal{D}$.
(1) Then $QT(A) = T(A)$;
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Theorem (EGLN)

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Let $A \in \mathcal{D}$.

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Let $A$ be a non-unital separable simple $C^*$-algebra which is in $\mathcal{D}$. Then $A$ has stable rank one.
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$).
$D^d$

**Definition**

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$,
Definition

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Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$, Suppose that, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$.
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Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$. Suppose that, for any $\epsilon > 0$, any finite subset $F \subset A$ and any $b \in A_+ \setminus \{0\}$, and any integer $n > 0$,
Definition

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Simple stably projectionless $C^*$-algebras  
Joint work with Guihua Gong Lafayette, Louisiana, Oct, 2017
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$, Suppose that, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $b \in A_+ \setminus \{0\}$, and any integer $n > 0$, there are $\mathcal{F}$-$\epsilon$-multiplicative c.p.c. linear maps $\phi: A \to A$ and $\psi: A \to D$ for some $C^*$-subalgebra $D \subset A$ such that $D \in C'_0$

\[ \|x - \text{diag}(\phi(x), \psi(x), \ldots, \psi(x))\| < \epsilon \text{ for all } x \in \mathcal{F} \cup \{a\}, \quad (\text{e 0.6}) \]
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$. Suppose that, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $b \in A_+ \setminus \{0\}$, and any integer $n > 0$, there are $\mathcal{F}$-$\epsilon$-multiplicative c.p.c. linear maps $\phi : A \to A$ and $\psi : A \to D$ for some $C^*$-subalgebra $D \subset A$ such that $D \in C'_0$

$$\|x - \text{diag}(\phi(x), \psi(x), ..., \psi(x))\| < \epsilon \text{ for all } x \in \mathcal{F} \cup \{a\}, \quad (e\ 0.6)$$

$$\phi(a) \precsim b, \quad (e\ 0.7)$$
Definition

Let $A$ be a non-unital and $\sigma$-unital simple $C^*$-algebra with $A = P(A)$ (Pedersen ideal of $A$) and with a strictly positive element $a \in A$ with $\|a\| = 1$ and with $\tau(a) > 3/4$ for all $\tau \in T(A)$. Suppose that, for any $\epsilon > 0$, any finite subset $F \subset A$ and any $b \in A_+ \setminus \{0\}$, and any integer $n > 0$, there are $F$-$\epsilon$-multiplicative c.p.c. linear maps $\phi : A \to A$ and $\psi : A \to D$ for some $C^*$-subalgebra $D \subset A$ such that $D \in C'_0$

\[ \|x - \text{diag}\left(\phi(x), \psi(x), \ldots, \psi(x)\right)\| < \epsilon \text{ for all } x \in F \cup \{a\}, \quad (e\ 0.6) \]

\[ \phi(a) \lesssim b, \]

\[ t\left(f_{1/4}(\psi(a))\right) \geq 3/4 \text{ for all } t \in T(D). \quad (e\ 0.7) \]
\( \mathcal{D}^d \)

**Definition**

Let \( A \) be a non-unital and \( \sigma \)-unital simple \( C^* \)-algebra with \( A = P(A) \) (Pedersen ideal of \( A \)) and with a strictly positive element \( a \in A \) with \( \|a\| = 1 \) and with \( \tau(a) > 3/4 \) for all \( \tau \in T(A) \), Suppose that, for any \( \epsilon > 0 \), any finite subset \( \mathcal{F} \subset A \) and any \( b \in A_+ \setminus \{0\} \), and any integer \( n > 0 \), there are \( \mathcal{F} \)-\( \epsilon \)-multiplicative c.p.c. linear maps \( \phi : A \to A \) and \( \psi : A \to D \) for some \( C^* \)-subalgebra \( D \subset A \) such that \( D \in C'_0 \)

\[
\| x - \text{diag}(\phi(x), \psi(x), \ldots, \psi(x)) \| < \epsilon \quad \text{for all } x \in \mathcal{F} \cup \{a\}, \quad (e0.6)
\]

\[
\phi(a) \lesssim b, \quad (e0.7)
\]

\[
t(f_{1/4}(\psi(a))) \geq 3/4 \quad \text{for all } t \in T(D). \quad (e0.8)
\]

Then then we say \( A \in \mathcal{D}^d \).
Theorem

Every $C^*$-algebra in $D_0$ are in $D^d$. 

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Simple stably projectionless $C^*$-algebras
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Theorem

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Every amenable $C^*$-algebra in $D$ is $\mathcal{Z}$-stable.
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Theorem

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For any unital $C^*$-algebra $A$, $CU(A)$ is the closure of commutator subgroup of $U(A)$.
**Theorem**

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**Theorem**

Every amenable $C^*$-algebra in $D$ is $\mathcal{Z}$-stable.

For any unital $C^*$-algebra $A$, $CU(A)$ is the closure of commutator subgroup of $U(A)$.

**Theorem**

Let $A \in D$ and $u \in CU(\tilde{A})$. Then

$$\text{cel}(u) \leq 5\pi.$$
Theorem

Let $A$ be a non-unital separable stably projectionless exact simple $C^*$-algebra with continuous scale which is $Z$-stable and $T(A) \neq \emptyset$. Then $K_0(\tilde{A})$ is weakly unperforated, i.e., if $x \in K_0(\tilde{A})$ with $kx \in K_0(\tilde{A}) + \{0\}$ for some integer $k \geq 1$, then $x \in K_0(\tilde{A}) + \{0\}$. Furthermore, if $p, q \in \mathcal{M}_s(\tilde{A})$ (for some $s \geq 1$) are two projections such that $\tau(q) < \tau(p)$ for all $\tau \in T(\tilde{A})$, then $q \precsim p$. 

Theorem

Let $A \in D_d$ and $B \in D$. Suppose that $\phi, \psi : A \to B$ are two homomorphisms which map strictly positive elements to strictly positive elements. Then $\phi$ and $\psi$ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(A, B)$, $\phi^T = \psi^T$ and $\phi^* = \psi^*$. 

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Simple stably projectionless $C^*$-algebras
Theorem

*Let* $A$ *be a non-unital separable stably projectionless exact simple* $C^*$-*algebra with continuous scale which is* $\mathbb{Z}$-*stable and* $T(A) \neq \emptyset$. 

Furthermore, if $p, q \in M_s(\tilde{A})$ *(for some* $s \geq 1$ *) are two projections such that* $\tau(q) < \tau(p)$ *for all* $\tau \in T(\tilde{A})$, *then* $q \preccurlyeq p$. 

**Theorem**

Let $A \in D_d$ and $B \in D_e$. *Suppose that* $\phi, \psi : A \to B$ *are two homomorphisms which map strictly positive elements to strictly positive elements.*

Then $\phi$ and $\psi$ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(A, B)$, $\phi^\tau = \psi^\tau$ and $\phi^\dag = \psi^\dag$. 

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Joint work with Guihua Gong
Lafayette, Louisiana, Oct, 2017

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Theorem

Let $A$ be a non-unital separable stably projectionless exact simple $C^*$-algebra with continuous scale which is $\mathcal{Z}$-stable and $T(A) \neq \emptyset$. Then $K_0(\tilde{A})$ is weakly unperforated, i.e.,

$$\ldots$$
Theorem

Let $A$ be a non-unital separable stably projectionless exact simple $C^*$-algebra with continuous scale which is $\mathbb{Z}$-stable and $T(A) \neq \emptyset$. Then $K_0(\tilde{A})$ is weakly unperforated, i.e., if $x \in K_0(\tilde{A})$ with $kx \in K_0(\tilde{A})_+ \setminus \{0\}$ for some integer $k \geq 1$, then $x \in K_0(\tilde{A})_+$. 

Furthermore, if $p, q \in M_s(\tilde{A})$ (for some $s \geq 1$) are two projections such that $\tau(q) < \tau(p)$ for all $\tau \in T(\tilde{A})$, then $q \ll p$. 

Theorem 

Let $A \in Dd$ and $B \in D$. Suppose that $\phi, \psi : A \to B$ are two homomorphisms which map strictly positive elements to strictly positive elements. Then $\phi$ and $\psi$ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(A, B)$, $\phi^T = \psi^T$ and $\phi^\dagger = \psi^\dagger$. 

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Simple stably projectionless $C^*$-algebras

Joint work with Guihua Gong Lafayette, Louisiana, Oct, 2017
Theorem

Let $A$ be a non-unital separable stably projectionless exact simple $C^*$-algebra with continuous scale which is $\mathcal{Z}$-stable and $T(A) \neq \emptyset$. Then $K_0(\tilde{A})$ is weakly unperforated, i.e., if $x \in K_0(\tilde{A})$ with $kx \in K_0(\tilde{A})_+ \setminus \{0\}$ for some integer $k \geq 1$, then $x \in K_0(\tilde{A})_+$. Furthermore, if $p, q \in M_s(\tilde{A})$ (for some $s \geq 1$) are two projections such that $\tau(q) < \tau(p)$ for all $\tau \in T(\tilde{A})$,
Theorem

Let $A$ be a non-unital separable stably projectionless exact simple $C^*$-algebra with continuous scale which is $\mathbb{Z}$-stable and $T(A) \neq \emptyset$. Then $K_0(\tilde{A})$ is weakly unperforated, i.e., if $x \in K_0(\tilde{A})$ with $kx \in K_0(\tilde{A})_+ \setminus \{0\}$ for some integer $k \geq 1$, then $x \in K_0(\tilde{A})_+$. Furthermore, if $p, q \in M_s(\tilde{A})$ (for some $s \geq 1$) are two projections such that $\tau(q) < \tau(p)$ for all $\tau \in T(\tilde{A})$, then $q \precsim p$. 

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Theorem

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Theorem

Let $A \in \mathcal{D}^d$ and $B \in \mathcal{D}$. 
Theorem

Let $A$ be a non-unital separable stably projectionless exact simple $C^*$-algebra with continuous scale which is $\mathbb{Z}$-stable and $T(A) \neq \emptyset$. Then $K_0(\tilde{A})$ is weakly unperforated, i.e., if $x \in K_0(\tilde{A})$ with $kx \in K_0(\tilde{A})_+ \setminus \{0\}$ for some integer $k \geq 1$, then $x \in K_0(\tilde{A})_+$. Furthermore, if $p, q \in M_s(\tilde{A})$ (for some $s \geq 1$) are two projections such that $\tau(q) < \tau(p)$ for all $\tau \in T(\tilde{A})$, then $q \lesssim p$.

Theorem

Let $A \in \mathcal{D}^d$ and $B \in \mathcal{D}$. Suppose that $\phi, \psi : A \to B$ are two homomorphisms which map strictly positive elements to strictly positive elements.
Theorem

Let $A$ be a non-unital separable stably projectionless exact simple $C^*$-algebra with continuous scale which is $\mathbb{Z}$-stable and $T(A) \neq \emptyset$. Then $K_0(\tilde{A})$ is weakly unperforated, i.e., if $x \in K_0(\tilde{A})$ with $kx \in K_0(\tilde{A})_+ \setminus \{0\}$ for some integer $k \geq 1$, then $x \in K_0(\tilde{A})_+$. Furthermore, if $p, q \in M_s(\tilde{A})$ (for some $s \geq 1$) are two projections such that $\tau(q) < \tau(p)$ for all $\tau \in T(\tilde{A})$, then $q \preceq p$.

Theorem

Let $A \in \mathcal{D}^d$ and $B \in \mathcal{D}$. Suppose that $\phi, \psi : A \to B$ are two homomorphisms which map strictly positive elements to strictly positive elements. Then $\phi$ and $\psi$ are approximately unitarily equivalent.
Theorem

Let $A$ be a non-unital separable stably projectionless exact simple $C^*$-algebra with continuous scale which is $\mathbb{Z}$-stable and $T(A) \neq \emptyset$. Then $K_0(\tilde{A})$ is weakly unperforated, i.e., if $x \in K_0(\tilde{A})$ with $kx \in K_0(\tilde{A})_+ \setminus \{0\}$ for some integer $k \geq 1$, then $x \in K_0(\tilde{A})_+$. Furthermore, if $p, q \in M_s(\tilde{A})$ (for some $s \geq 1$) are two projections such that $\tau(q) < \tau(p)$ for all $\tau \in T(\tilde{A})$, then $q \precsim p$.

Theorem

Let $A \in \mathcal{D}^d$ and $B \in \mathcal{D}$. Suppose that $\phi, \psi : A \to B$ are two homomorphisms which map strictly positive elements to strictly positive elements. Then $\phi$ and $\psi$ are approximately unitarily equivalent if and only if

$$[\phi] = [\psi] \text{ in } KL(A, B),$$

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Theorem

Let $A \in \mathcal{D}_d$ and $B \in \mathcal{D}$. Suppose that $\phi, \psi : A \to B$ are two homomorphisms which map strictly positive elements to strictly positive elements. Then $\phi$ and $\psi$ are approximately unitarily equivalent if and only if

$$[\phi] = [\psi] \text{ in } KL(A, B),$$
$$\phi_T = \psi_T \text{ and } \phi^\dagger = \psi^\dagger$$

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Simple stably projectionless $C^*$-algebras
Joint work with Guihua Gong
Lafayette, Louisiana, Oct, 2017
Theorem

Let $A \in \mathcal{D}^d$ and $B \in \mathcal{D}$. Suppose that $\phi, \psi : A \to B$ are two homomorphisms which maps strictly positive elements to strictly positive elements.

Then $\phi$ and $\psi$ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $\text{KL}(A, B)$, $\phi^T = \psi^T$ and $\phi^\dagger = \psi^\dagger$. 

$\phi^T$, $\psi^T$ are affine continuous maps from $T(B)$ to $T(A)$ induced by $\phi$ and $\psi$, respectively. $\phi^\dagger$, $\psi^\dagger : \tilde{U}(\tilde{A})/\text{CU}(\tilde{A}) \to \tilde{U}(\tilde{B})/\text{CU}(\tilde{B})$ are the continuous homomorphisms given by $\phi$ and $\psi$. 

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Simple stably projectionless C*-algebras
Theorem

Let $A \in \mathcal{D}^d$ and $B \in \mathcal{D}$. Suppose that $\phi, \psi : A \to B$ are two homomorphisms which maps strictly positive elements to strictly positive elements. Then $\phi$ and $\psi$ are approximately unitarily equivalent if and only if

$$[\phi] = [\psi] \text{ in } KL(A, B), \phi_T = \psi_T \text{ and } \phi^\dagger = \psi^\dagger.$$
Theorem

Let $A \in \mathcal{D}^d$ and $B \in \mathcal{D}$. Suppose that $\phi, \psi : A \to B$ are two homomorphisms which maps strictly positive elements to strictly positive elements. Then $\phi$ and $\psi$ are approximately unitarily equivalent if and only if

\[ [\phi] = [\psi] \text{ in } KL(A, B), \phi_T = \psi_T \text{ and } \phi^\dagger = \psi^\dagger. \]

$\phi_T$, $\psi_T$ are affine continuous maps from $T(B)$ to $T(A)$ induced by $\phi$ and $\psi$, respectively.

$\phi^\dagger$, $\psi^\dagger : U(\tilde{A})/CU(\tilde{A}) \to U(\tilde{B})/CU(\tilde{B})$ are the continuous homomorphisms given by $\phi$ and $\psi$. 
Theorem

Let $A$ and $B$ be two separable amenable $C^*$-algebras in $D^d$ which satisfies the UCT.

Consider the case that $A$ and $B$ have continuous scale. Then $\text{Ell}(A) \sim = \text{Ell}(B)$ means the following:

Two isomorphisms $\kappa_i : K_i(A) \to K_i(B)$, $i = 0, 1$ one affine homeomorphism $\kappa_T : T(A) \to T(B)$; Moreover, for all $\tau \in T(B)$ and $x \in K_0(A)$,

$$\kappa_0(x)(\tau) = \kappa_{-1}T(\tau)(x).$$
Theorem

Let $A$ and $B$ be two separable amenable $C^*$-algebras in $D^d$ which satisfies the UCT. Then $A \cong B$ if and only if

$$\text{Ell}(A) \cong \text{Ell}(B).$$

Consider the case that $A$ and $B$ have continuous scale. Then $\text{Ell}(A) \cong \text{Ell}(B)$ means the following:

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Simple stably projectionless $C^*$-algebras

Joint work with Guihua Gong
Lafayette, Louisiana, Oct, 2017
Theorem

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Two isomorphisms $\kappa_i : K_i(A) \to K_i(B), i = 0, 1$

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Two isomorphisms $\kappa_i : K_i(A) \to K_i(B)$, $i = 0, 1$

one affine homeomorphism $\kappa_T : T(A) \to T(B)$;

Moreover, for all $\tau \in T(B)$ and $x \in K_0(A)$,
Theorem

Let $A$ and $B$ be two separable amenable $C^*$-algebras in $\mathcal{D}^d$ which satisfies the UCT. Then $A \cong B$ if and only if

$$\text{Ell}(A) \cong \text{Ell}(B).$$

Consider the case that $A$ and $B$ have continuous scale. Then $\text{Ell}(A) \cong \text{Ell}(B)$ means the following:

Two isomorphisms $\kappa_i : K_i(A) \to K_i(B)$, $i = 0, 1$

one affine homeomorphism $\kappa_T : T(A) \to T(B)$;

Moreover, for all $\tau \in T(B)$ and $x \in K_0(A)$,

$$\kappa_0(x)(\tau) = \kappa_T^{-1}(\tau)(x).$$
Theorem

Let $A \in D$. Then $A \in D_0$ if and only if $\ker \rho_A = K_0(A)$. 

Corollary

Let $A$ and $B$ be two separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT. Then $A \otimes Z \sim B \otimes Z$ if and only if $\Ell(A \otimes Z) \sim \Ell(B \otimes Z)$. 

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Simple stably projectionless $C^*$-algebras
**Theorem**

Let $A \in D$. Then $A \in D_0$ if and only if $\ker \rho_A = K_0(A)$.

**Theorem**

Let $A$ and $B$ be two stably projectionless simple amenable $C^*$-algebras with UCT.
Theorem

Let $A \in \mathcal{D}$. Then $A \in \mathcal{D}_0$ if and only if $\ker \rho_A = K_0(A)$.

Theorem

Let $A$ and $B$ be two stably projectionless simple amenable $C^*$-algebras with UCT such that $gTR(A) \leq 1$, $gTR(B) \leq 1$. 

Corollary

Let $A$ and $B$ be two separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT.

Then $A \otimes \mathbb{Z}_0 \sim B \otimes \mathbb{Z}_0$ if and only if $\text{Ell}(A \otimes \mathbb{Z}_0) \sim \text{Ell}(B \otimes \mathbb{Z}_0)$. 

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Simple stably projectionless $C^*$-algebras
Theorem

Let $A \in \mathcal{D}$. Then $A \in \mathcal{D}_0$ if and only if $\ker \rho_A = K_0(A)$.

Theorem

Let $A$ and $B$ be two stably projectionless simple amenable $C^*$-algebras with UCT such that $gTR(A) \leq 1$, $gTR(B) \leq 1$. Suppose that $\ker \rho_A = K_0(A)$ and $\ker \rho_B = K_0(B)$.
Theorem

Let $A \in \mathcal{D}$. Then $A \in \mathcal{D}_0$ if and only if $\ker \rho_A = K_0(A)$.

Theorem

Let $A$ and $B$ be two stably projectionless simple amenable $C^*$-algebras with UCT such that $gTR(A) \leq 1$, $gTR(B) \leq 1$. Suppose that $\ker \rho_A = K_0(A)$ and $\ker \rho_B = K_0(B)$. Then $A \cong B$ if and only if $\Ell(A) \cong \Ell(B)$.

Corollary

Let $A$ and $B$ be two separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT. Then $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$ if and only if $\Ell(A \otimes \mathcal{Z}) \cong \Ell(B \otimes \mathcal{Z})$.
Theorem

Let $A \in D$. Then $A \in D_0$ if and only if $\ker \rho_A = K_0(A)$.

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Let $A$ and $B$ be two stably projectionless simple amenable $C^*$-algebras with UCT such that $gTR(A) \leq 1$, $gTR(B) \leq 1$. Suppose that $\ker \rho_A = K_0(A)$ and $\ker \rho_B = K_0(B)$. Then $A \cong B$ if and only if $\Ell(A) \cong \Ell(B)$.

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Let $A$ and $B$ be two separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT.
**Theorem**

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**Theorem**

Let $A$ and $B$ be two stably projectionless simple amenable $C^*$-algebras with UCT such that $gTR(A) \leq 1$, $gTR(B) \leq 1$. Suppose that $\ker \rho_A = K_0(A)$ and $\ker \rho_B = K_0(B)$. Then $A \cong B$ if and only if $\mathcal{E}(A) \cong \mathcal{E}(B)$.

**Corollary**

Let $A$ and $B$ be two separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT. Then $A \otimes \mathbb{Z}_0 \cong B \otimes \mathbb{Z}_0$.
Theorem

Let $A \in \mathcal{D}$. Then $A \in \mathcal{D}_0$ if and only if $\ker \rho_A = K_0(A)$.

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Let $A$ and $B$ be two stably projectionless simple amenable $C^*$-algebras with UCT such that $gTR(A) \leq 1$, $gTR(B) \leq 1$. Suppose that $\ker \rho_A = K_0(A)$ and $\ker \rho_B = K_0(B)$. Then $A \cong B$ if and only if

$$\text{Ell}(A) \cong \text{Ell}(B).$$

Corollary

Let $A$ and $B$ be two separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT. Then $A \otimes \mathcal{Z}_0 \cong B \otimes \mathcal{Z}_0$ if and only if

$$\text{Ell}(A \otimes \mathcal{Z}_0) \cong \text{Ell}(B \otimes \mathcal{Z}_0).$$
Theorem

Let $A$ and $B$ be two stably projectionless separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT.

Suppose also that $K_0(A) = Tor(K_0(A))$ and $K_0(B) = Tor(K_0(B))$.

Then $A \sim B$ if and only if $Ell(A) \sim Ell(B)$.
Theorem

Let $A$ and $B$ be two stably projectionless separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT. Suppose also that $K_0(A) = \text{Tor}(K_0(A))$ and $K_0(B) = \text{Tor}(K_0(B))$. Then $A \cong B$ if and only if $\text{Ell}(A) \cong \text{Ell}(B)$. 
Theorem

Let $A$ and $B$ be two stably projectionless separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT. Suppose also that $K_0(A) = \text{Tor}(K_0(A))$ and $K_0(B) = \text{Tor}(K_0(B))$. Then $A \cong B$.
Theorem

Let $A$ and $B$ be two stably projectionless separable simple $C^*$-algebras with finite nuclear dimension and satisfy UCT. Suppose also that $K_0(A) = \text{Tor}(K_0(A))$ and $K_0(B) = \text{Tor}(K_0(B))$. Then $A \cong B$ if and only if

$$\text{Ell}(A) \cong \text{Ell}(B).$$
Note every $C^*$-algebra $A \in \mathcal{D}$ is in $\mathcal{D}^d$. 
Note every $C^*$-algebra $A \in D$ is in $D^d$. 

Let $D$ be a non-unital $C^*$-algebra. Denote by $D_T$ the $C^*$-subalgebra of $C(T, \tilde{D})$ generated by $C_0(T) \{1\} \otimes 1_{\tilde{D}}$ and $1_{C(T)} \otimes D$. 

The unitization of $D_T$ is $C(T, \tilde{D}) = C(T) \otimes \tilde{D}$.

Let $C$ be another non-unital $C^*$-algebra, $L : D_T \to C$ be a c.p.c. map and $L_{\sim} : C(T) \otimes \tilde{D} \to \tilde{C}$ be the unitization.

Denote by $z$ the standard unitary generator of $C(T)$.

For any finite subset $F \subset C(T) \otimes \tilde{D}$, any finite subset $F_d \subset \tilde{D}$, and $\epsilon > 0$, there exists a finite subset $G \subset D$ and $\delta > 0$ such that, whenever $\phi : D \to C$ is a $G_\delta$-multiplicative c.p.c. map (for any $C^*$-algebra $C$) and $\|u \cdot \phi(g)\| < \delta$ for some unitary $u \in \tilde{C}$ and for all $g \in G$, there exists a $F_\epsilon$-multiplicative c.p.c. map $L_1 : C(T) \otimes \tilde{D} \to \tilde{C}$ such that $\|L_1(z \otimes 1) - u\| < \epsilon$ and $\|L_1(1 \otimes d) - \phi_{\sim}(d)\| < \epsilon$ for all $d \in F_d$.

We will denote such $L_1$ by $\Phi_{\phi, u}$. 

Note every $C^*$-algebra $A \in D$ is in $D^d$. 

Let $D$ be a non-unital $C^*$-algebra. Denote by $D_T$ the $C^*$-subalgebra of $C(T, \tilde{D})$ generated by $C_0(T) \{1\} \otimes 1_{\tilde{D}}$ and $1_{C(T)} \otimes D$. 

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Let $C$ be another non-unital $C^*$-algebra, $L : D^\mathbb{T} \to C$ be a c.p.c. map and $L^\sim : C(\mathbb{T}) \otimes \tilde{D} \to \tilde{C}$ be the unitization.
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Let $C$ be another non-unital $C^*$-algebra, $L : D^\mathbb{T} \to C$ be a c.p.c. map and $L^\sim : C(\mathbb{T}) \otimes \tilde{D} \to \tilde{C}$ be the unitization. Denote by $z$ the standard unitary generator of $C(\mathbb{T})$. For any finite subset $\mathcal{F} \subset C(\mathbb{T}) \otimes \tilde{D}$, any finite subset $\mathcal{F}_d \subset \tilde{D}$, and $\epsilon > 0$, 
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Note every $C^*$-algebra $A \in \mathcal{D}$ is in $\mathcal{D}^d$. Let $D$ be a non-unital $C^*$-algebra. Denote by $D^\mathbb{T}$ the $C^*$-subalgebra of $C(\mathbb{T}, \tilde{D})$ generated by $C_0(\mathbb{T} \setminus \{1\}) \otimes 1_{\tilde{D}}$ and $1_{C(\mathbb{T})} \otimes D$. The unitization of $D^\mathbb{T}$ is $C(\mathbb{T}, \tilde{D}) = C(\mathbb{T}) \otimes \tilde{D}$.

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$$\|L_1(z \otimes 1) - u\| < \epsilon \quad \text{and} \quad \|L_1(1 \otimes d) - \phi^\sim(d)\| < \epsilon \quad \text{for all} \quad d \in \mathcal{F}_d.$$
Note every $C^*$-algebra $A \in \mathcal{D}$ is in $\mathcal{D}^d$. Let $D$ be a non-unital $C^*$-algebra. Denote by $D^\mathbb{T}$ the $C^*$-subalgebra of $C(\mathbb{T}, \tilde{D})$ generated by $C_0(\mathbb{T} \setminus \{1\}) \otimes 1_{\tilde{D}}$ and $1_{C(\mathbb{T})} \otimes D$. The unitization of $D^\mathbb{T}$ is $C(\mathbb{T}, \tilde{D}) = C(\mathbb{T}) \otimes \tilde{D}$.

Let $C$ be another non-unital $C^*$-algebra, $L : D^\mathbb{T} \to C$ be a c.p.c. map and $L^\sim : C(\mathbb{T}) \otimes \tilde{D} \to \tilde{C}$ be the unitization. Denote by $z$ the standard unitary generator of $C(\mathbb{T})$. For any finite subset $\mathcal{F} \subset C(\mathbb{T}) \otimes \tilde{D}$, any finite subset $\mathcal{F}_d \subset \tilde{D}$, and $\epsilon > 0$, there exists a finite subset $\mathcal{G} \subset D$ and $\delta > 0$ such that, whenever $\phi : D \to C$ is a $\mathcal{G}$-$\delta$ -multiplicative c.p.c. map (for any $C^*$-algebra $C$) and $\|[u, \phi(g)]\| < \delta$ for some unitary $u \in \tilde{C}$ and for all $g \in \mathcal{G}$, there exists a $\mathcal{F}$-$\epsilon$-multiplicative c.p.c. map $L_1 : C(\mathbb{T}) \otimes \tilde{D} \to \tilde{C}$ such that

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We will denote such $L_1$ by $\Phi_{\phi, u}$. 
Lemma

Let $A \in \mathcal{D}^d$ have continuous scale. For any $1 > \epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, $\sigma > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\{p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k\}$ of projections of $M_N(\tilde{A})$ (for some integer $N \geq 1$) such that $\{[p_1] - [q_1], [p_2] - [q_2], \ldots, [p_k] - [q_k]\}$ generates a free subgroup $G_u$ of $K_0(A)$, and a finite subset $\mathcal{P} \subset K(A)$, satisfying the following:

Suppose that $\phi : A \to B \otimes U$ is a homomorphism which maps strictly positive elements to strictly positive elements, where $B \in \mathcal{D}$ has continuous scale and $U$ is a UHF-algebra of infinite type. If $u \in U(B \otimes U)$ is a unitary such that

$$||[\phi(x), u]] < \delta \text{ for all } x \in \mathcal{G},$$
$$\text{Bott}(\phi, u)|_\mathcal{P} = 0,$$
$$\text{dist}(\langle((1 - \phi^\sim(p_i)) + \phi^\sim(p_i)u)(1 - \phi^\sim(q_i)) + \phi^\sim(q_i)u^*\rangle, \bar{1}) < \sigma \text{ and}$$
$$\text{dist}(\bar{u}, \bar{1}) < \sigma,$$

(where $u = u \otimes 1_{M_N}$), then there exists a continuous path of unitaries
Lemma

(continue) Suppose that $\phi : A \to B \otimes U$ is a homomorphism which maps strictly positive elements to strictly positive elements, where $B \in \mathcal{D}$ has continuous scale and $U$ is a UHF-algebra of infinite type. If $u \in U(B \otimes U)$ is a unitary such that

$$\| [\phi(x), u] \| < \delta \text{ for all } x \in \mathcal{G},$$

$$\text{Bott}(\phi, u)|_P = 0,$$

$$\text{dist}(\langle ((1 - \phi^*(p_i)) + \phi^*(p_i)u)(1 - \phi^*(q_i)) + \phi^*(q_i)u^* \rangle, \tilde{1}) < \sigma \text{ and }$$

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(weeks $u = u \otimes 1_{M_N}$), then there exists a continuous path of unitaries

$$\{u(t) : t \in [0, 1]\} \subset U_0(B \otimes U)$$

such that

$$u(0) = u, \ u(1) = 1 \quad (e\ 0.12)$$

$$\| [\phi(a), u(t)] \| < \epsilon \text{ for all } a \in \mathcal{F} \text{ and for all } t \in [0, 1]. \quad (e\ 0.13)$$
Theorem

Let $A \in \mathcal{D}^d$ with continuous scale which satisfies the UCT. Then, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with \(\{g_1, g_2, \ldots, g_k\} = \mathcal{P} \cap K_1(A)\), there exists $\eta > 0$ and a finite subset $\mathcal{Q} \subset K(A)$ satisfy the following: if $\alpha \in KL(A^\mathbb{T}, B)$, where $B = C \otimes U$, $C \in \mathcal{D}$ is amenable simple $C^*$-algebra with continuous scale and $U$ is a UHF-algebra of infinite type such that

\[
|\rho_B(\beta(\alpha(g_i)))(\tau)| < \eta \quad \text{for all } \tau \in T(B), \quad i = 1, 2, \ldots, k, \quad (e\,0.14)
\]

and if $\phi : A \to B$ is homomorphism which maps strictly positive elements to strictly positive elements, there exists a unitary $u \in CU(\tilde{B})$ such that

\[
\|[\phi(a), u]\| < \epsilon \quad \text{for all } a \in \mathcal{F}, \quad (e\,0.15)
\]

\[
\text{Bott}(\phi, u)|_\mathcal{P} = \alpha(\beta)|_\mathcal{P}. \quad (e\,0.16)
\]
Theorem

Let $C_1$ be a simple $C^*$-algebra in $D$ with continuous scale which satisfies the UCT,
Theorem

Let $C_1$ be a simple $C^*$-algebra in $\mathcal{D}$ with continuous scale which satisfies the UCT, let $A_1$ be a separable simple $C^*$-algebra in $\mathcal{D}$ with continuous scale, and let $U_1$ and $U_2$ be two UHF-algebras of infinite type. Let $C = C_1 \otimes U_1$ and $A = A_1 \otimes U_2$.

Suppose that $\phi_1, \phi_2 : C \to A$ are two monomorphisms which maps strictly positive elements to strictly positive elements. Then they are asymptotically unitarily equivalent if and only if $[\phi_1] = [\phi_2]$ in $KK(\mathcal{C}, \mathcal{A})$.
**Theorem**

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Simple stably projectionless $C^*$-algebras

Joint work with Guihua Gong
Lafayette, Louisiana
Oct, 2017
**Theorem**

Let $C_1$ be a simple $C^*$-algebra in $D$ with continuous scale which satisfies the UCT, let $A_1$ be a separable simple $C^*$-algebra in $D$ with continuous scale, and let $U_1$ and $U_2$ be two UHF-algebras of infinite type. Let $C = C_1 \otimes U_1$ and $A = A_1 \otimes U_2$. Suppose that $\phi_1, \phi_2 : C \to A$ are two monomorphisms which maps strictly positive elements to strictly positive elements. Then they are asymptotically unitarily equivalent if and only if $[\phi_1] = [\phi_2]$ in $KK(C, A)$, $\phi_1^\wedge = \phi_2^\wedge$, $(\phi_1)^T = (\phi_2)^T$, and $R_{\phi_1, \phi_2} = 0$. 

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$$[\phi_1] = [\phi_2] \text{ in } KK(C, A),$$

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\[
[\phi_1] = [\phi_2] \quad \text{in} \quad KK(C, A),
\]

\[
\phi^\dagger = \psi^\dagger, \quad (\phi_1)^T = (\phi_2)^T
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Simple stably projectionless $C^*$-algebras

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Theorem

Let $A$ and $B$ be two stably projectionless separable amenable simple $C^*$-algebras which satisfy the UCT. 

$g_{TR}(A) \leq 1$ and $g_{TR}(B) \leq 1$. Then $A \sim B$ if and only if $Ell(A) \sim Ell(B)$. 

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