CBMS LECTURE SERIES

WILHELM WINTER

ABSTRACT. Notes taken by Stuart White during the lectures and **not proof read**. Errors will abound.

1. NUCLEARITY: APPROXIMATION AND PERTURBATION

Recall:

Definition 1.1. A C^*-algebra.

Definition 1.2. A cp map

Theorem 1.3. Stinespring's theorem. Every cp map is a compression of a *-homomorphism.

Definition 1.4. $\phi : A \to B$ cp has *order zero* if $a \perp b \implies \phi(a) \perp \phi(b)$. [1]

Theorem 1.5. (i) $\phi : A \to B$ has order zero iff

$$\phi(\cdot) = h\pi(\cdot) = \pi(\cdot)h,$$

where $\pi : A \to M(C^*(\phi(A))) \subseteq B^{**}$ is a *-hm and $h \in M(C^*(\phi(A)))_+ \cap \phi(A)'$. When A is unital $h = \phi(1_A)$.

(ii) $CPC_{\text{ord0}}(A, B) \leftrightarrow \text{Hom}(C_0((0, 1], A), B))$ is a one-one correspondence.

(iii) $\phi : A \to B$ be cp order zero, $f \in C_0(0, \|\phi\|]$, then we can define a new cp order zero map $f(\phi) : A \to B$ by $f(\phi)(a) = f(h)\pi(a)$. This gives a functional calculus to order zero maps.

Order zero maps are precisely those compressions of *-hms by positive elements which commute with the range. A unital order zero map is just a *-homomorphism. The bijection in (ii) arises as $C_0((0, 1]$ is the universal C^*-algebra generated by a positive contraction (this corresponds to the h).

Recall:

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[1]Positive elements a, b are orthogonal if $ab = 0$.

Theorem 1.6 (Choi - Effros)**.** A C^*-algebra A is nuclear iff A has the completely positive approximation property (CPAP).

Theorem 1.7 (Hirschberg-Kirchberg-White)**.** A is nuclear iff A has cpc approximation

$$\begin{array}{ccc} A & \xrightarrow{\quad \mathrm{id} \quad} & A \\ & {\scriptstyle \psi_\lambda} \searrow \qquad \nearrow {\scriptstyle \phi_\lambda} & \\ & F_\lambda & \end{array}$$

with each ϕ_λ a convex combination of cpc order zero maps.

Proof uses: injective implies hyperfinite, projectivity of cp order zero maps with finite dimensional domains, Hahn-Banach.

Exercise 1.8. What happens with bounded number of order zero maps and convex combinations? Does this characterise AF algebras?

Elliott program: Classify nuclear C^*-algebras by K theoretic data.

Notes:

- There are good reasons to use K-theory as an invariant.
- Need to enrich K-theory with trace spaces.
- K-theory not quite compatible with CPAP.
- K-theory stable under small perturbations.

Theorem 1.9 (Christensen)**.** Separable AF algebras are stable under small perturbations.

Theorem 1.10 (CSSWW)**.** Separable nuclear algebras are stable under small perturbations.

2. Strongly self absorbing C^*-algebras

Definition 2.1 (Toms-W)**.** $\mathcal{D} \neq \mathbb{C}$ separable and unital is *strongly self absorbing* (ssa) if there is

$$\phi : \mathcal{D} \xrightarrow{\cong} \mathcal{D} \otimes \mathcal{D}$$

such that $\phi \approx_{au} \mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$.[2]

Proposition 2.2. *A separable unital $\mathcal{D} \neq \mathbb{C}1$ is ssa if $\mathcal{D} \otimes \mathcal{D}^{\otimes k}$ ($\cong \mathcal{D}^{\otimes \infty}$) for some $k > 1$ and $\mathcal{D}$ has approximately inner half flip (i.e. the first factor embedding and the second factor embedding are approximately inner equivalent: $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}} \approx_{au} 1_{\mathcal{D}} \otimes \mathrm{id}_{\mathcal{D}}$).*

[2] A few remarks, we could have interchanged the roles of the first and second factor embedding, and we used the minimal tensor product. We could have used another tensor product, this is equivalent as it turns out that this condition implies nuclearity.

Proof of $\mathcal{D} \cong \mathcal{D}^{\otimes\infty}$ uses a one sided Elliott intertwining.

Corollary 2.3. $\mathcal{D}$ ssa, then any unital endomorphism of $\mathcal{D}$ is approximately inner via unitaries which are trivial in K_1.

Remark 2.4. In fact any unital endomorphism is strongly asymptotically inner (i.e. there is a continuous map of unitaries starting at the unit connecting the unital endomorphism to the identity). This uses K_1-injectivity[3], and an argument of Dardarlat-W.

Theorem 2.5 (Effros-Rosenberg)**.** If $\mathcal{D}$ ssa, then $\mathcal{D}$ is simple and nuclear (in fact we only need approximately inner half flip for this).

Proof. Suppose $u_n \approx c_n = \sum_{j=1}^{m_n} x_{n,j} \otimes y_{n,j} \in \mathcal{D} \odot \mathcal{D}$ implements the half flip (where each $x_{n,j}$, $y_{n,j}$ are contactions). Choose $\phi \in S(\mathcal{D})$ and define $T_n : \mathcal{D} \to \mathcal{D}$ cpc by

$$T_n(d) = (\phi \otimes \mathrm{id}_{\mathcal{D}})(c_n(d \otimes 1_{\mathcal{D}})c_n^*).$$

These have finite rank as c_n is a finite linear combination. Indeed:

$$T_n(\mathcal{D}) \subset \mathrm{Span}(\{\mathbb{C} \otimes y_{n,j}y_{n,k}^* : j, k \leq m_n).$$

Further $T_n \overset{n\to\infty}{\to} \mathrm{id}_{\mathcal{D}}$ in point norm, so $\mathcal{D}$ has the CPAP.

For simplicity, suppose $J \trianglelefteq \mathcal{D}$. then $J \otimes \mathcal{D}, \mathcal{D} \otimes J \trianglelefteq \mathcal{D} \otimes \mathcal{D}$. Then

$$J \otimes \mathcal{D} \subseteq \overline{\bigcup_{n=1}^{\infty} u_n(\mathcal{D} \otimes J)u_n^*} \subseteq \mathcal{D} \otimes J.$$

By symmetry $\mathcal{D} \otimes J \subseteq J \otimes \mathcal{D}$ and hence $\mathcal{D} \otimes J = J \otimes \mathcal{D}$. An easy argument with states can be used to reach a contradiction if J is non-trivial. □

Theorem 2.6 (Kirchberg)**.** If $\mathcal{D}$ is ssa, then $\mathcal{D}$ is either purely infinite or stably finite with unique trace.

Outline of the dichotomy statement[4]. Suppose $\mathcal{D}$ is not stably finite[5] so $\mathcal{K} \subseteq \mathcal{T} \subseteq M_r \otimes \mathcal{D}$ for some $r \in \mathbb{N}$. We need to show that for any $0 \neq d \in \mathcal{D}_+$, there is a subalgebra of $\overline{d\mathcal{D}d} \otimes \mathcal{D}^{\otimes(r+1)}$ (from there we use results of Blackadar and Cuntz to reach pure infiniteness). Choose non

[3]i.e. the map from the unitaries modulo the connected component into K_1 is injective.

[5]i.e. some matrix algebra over $\mathcal{D}$ contains an infinite projection.

zero pairwise orthogonal positive elements $e_1, \ldots, e_r \in \mathcal{D}_+$ and define

$$f_1 = d \otimes e_1 \otimes \cdots \otimes e_r \otimes 1_{\mathcal{D}} \in \overline{d\mathcal{D}d} \otimes \mathcal{D}^{\otimes(r+1)} \subset \mathcal{D} \otimes \mathcal{D}^{\otimes(r+1)}$$
$$f_2 = d \otimes e_r \otimes e_1 \otimes \cdots \otimes e_{r-1} \otimes 1_{\mathcal{D}} \in \overline{d\mathcal{D}d} \otimes \mathcal{D}^{\otimes(r+1)} \subset \mathcal{D} \otimes \mathcal{D}^{\otimes(r+1)}$$
$$\ldots$$
$$f_r = d \otimes e_2 \otimes e_3 \otimes \cdots \otimes e_1 \otimes 1_{\mathcal{D}} \in \overline{d\mathcal{D}d} \otimes \mathcal{D}^{\otimes(r+1)} \subset \mathcal{D} \otimes \mathcal{D}^{\otimes(r+1)}.$$

These are all approximately inner equivalent (using the approximately inner flip). These can be used to define a non-zero *-hm

$$\Phi : C_0(0,1] \otimes M_r \otimes \mathcal{D} \to \overline{d\mathcal{D}d} \otimes \mathcal{D}^{\otimes(r+1)}.$$

(as the resulting $f_1, \ldots, f_r$ are pairwise orthogonal). Since $M_r \otimes \mathcal{D}$ contains the compact this gives the required stable subalgebra. ☐

Theorem 2.7. Let $A, \mathcal{D}$ be separable and $\mathcal{D}$ be ssa. TFAE:

(i) A is $\mathcal{D}$-stable, i.e. $A \cong A \otimes \mathcal{D}$;
(ii) $\exists \rho : \mathcal{D} \to M(A)_\omega \cap A'$ unital *-hm;
(iii) $\exists \sigma : A \otimes \mathcal{D} \to A_\omega$ *-hm such that $\sigma \circ (\mathrm{id}_A \otimes 1_{\mathcal{D}}) = \iota_A$.

Proposition 2.8 (One-sided intertwining)**.** *Let A, B be separable $\phi : A \to B$ injective $(v_n)_n \subset (\widetilde{B})_\omega \cap \phi(A)'$ unitaries such that $d(v_n^* b v_n, \phi(A)_\omega) \overset{n\to 0}{\to} 0$ for $b \in B$. Then $A \cong B$ (in fact we get an isomorphism which is approximately unitarily equivalent to ϕ).*

This proposition is used to prove (ii) implies (i).

Corollary 2.9. Permanence of $\mathcal{D}$-stability , i.e. $\mathcal{D}$-stability passes to hereditary subalgebras, quotients, extensions (harder).

Corollary 2.10. If $\mathcal{D} = \lim_\to \mathcal{D}_k$ ssa, then A is $\mathcal{D}$ stable iff $\mathcal{D}_k \hookrightarrow M(A)_\omega \cap A'$.

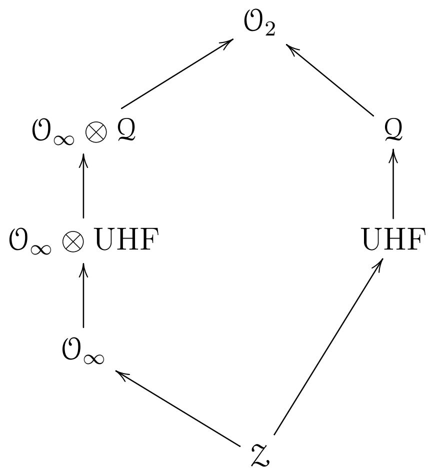
Examples 2.11. of strongly self absorbing algebras:

- UHF algebras of infinite type.
- $\mathcal{Z} = \lim_\to \mathcal{Z}_{p_k,q_k}$ where p_k, q_k are relatively prime, simple and unique trace, and $\mathcal{Z}_{p,q} = \{f : C([0,1], M_p \otimes M_q) : f(0) \in M_p \otimes 1_q,\ f(1) \in 1_p \otimes M_q\}$.
- $\mathcal{O}_\infty$
- $\mathcal{O}_2$
- $\mathcal{O}_\infty \otimes$ UHF of infinite type.

Remarks 2.12. (i) Can construct these using generators and relations. Show that ssa uses classification of some sort.

(ii) $\mathcal{O}_2$ is the uniquely determined object in the category of ssa algebras (with au classes of *-hms of maps). Proved by Kirchberg's embedding theorem.

(iii) $\mathcal{O}_2$ is the ssa C^*-algebra with trivial K_0.
(iv) $\mathcal{Q}$ (the universal UHF algebra) can be interpreted as a final object in a suitable sub category of the ssa algebras. It's the ssa C^*-algebra with $K_0(\mathcal{Q}) = \mathbb{Q}$) and finite decomposition rank.



Question 2.13. Is there an instrinsic characterisation of $\mathcal{O}_\infty$ or $\mathcal{Z}$? (The UCT is not allowed).

3. Topological dimension

Definition 3.1. X compact metrisable, $\dim X \leq n$ iff for a finite open cover $\mathcal{V}$ there is a open cover $\mathcal{U} = (U_k^{(i)})_{i=0,\ldots,n,\ k=1,\ldots K^{(i)}}$ such that $\mathcal{U} \precsim \mathcal{V}$ (i.e. each element of $\mathcal{U}$ is contained in an element of $\mathcal{V}$) and $U(i)_k \cap U_{k'}^{(i)} = \emptyset$ for $k \neq k'$.

Definition 3.2. Let A be a C^*-algebra. Say $\dim_{\mathrm{nuc}}(A) \leq n$ (the nuclear dimension) if, for every $\mathcal{F} \subset\subset A$ and $\varepsilon > 0$ there is

$$A \xrightarrow{\psi} F = F^{(0)} \oplus \cdots \oplus F^{(n)} \xrightarrow{\phi = \sum \phi^{(i)}} A$$

such that

- F is finite dimensional;
- ψ is cpc;
- each $\phi^{(i)}$ is cpc order zero;
- $\phi\psi \approx_{\mathcal{F},\varepsilon} \mathrm{id}_A$.

We say that $\mathrm{dr}(A) \leq n$ (the decomposition rank) if in addition ϕ can be taken cpc.

Proposition 3.3. $\mathrm{dr}(C_0(X)) = \dim_{\mathrm{nuc}}(C_0(X)) = \dim X$. *These notions have permanance properties (at least at the level of having finite* *decomposition rank and finite nuclear dimension) such as quotients, hereditary subalgebras, Mortia equivalence. Finite nuclear dimension is invariant under extensions: decomposition rank is not.*

Proposition 3.4. $\mathcal{F} \subset\subset A_+^1$, $\eta > 0$.

(i) *If* $\mathrm{dr}(A) \leq n$, *and* (F, ψ, ϕ) *be an* n*-decomposable cpc approximation for* $\mathcal{F} \cup \mathcal{F}^2$ *upto* η*. Then, for any central projection* $p \in F^{(i)}$*, then*
$$a\phi(p) \approx_{4\eta^{1/2}} \phi(\psi(a)p), \quad \forall a \in \mathcal{F}.$$
In particular
$$a\phi^{(i)}(1_{F^{(i)}}) \approx \phi^{(i)}(\psi^{(i)}(a)), \quad \forall a \in \mathcal{F}.$$
This enables us to cut out the i*-th order zero part of the approximation.*

(ii) *If* $\dim_{\mathrm{nuc}}(A) \leq n$, *and* (F, ψ, ϕ) *be an* n*-decomposable cp approximation for* $\mathcal{F} \cup \mathcal{F}^2$ *upto* η*. Then,*
$$a\phi^{(i)}(\psi^{(i)}(1_A)) \approx \phi^{(i)}\psi^{(i)}(a), \quad \forall a \in \mathcal{F}.$$

Proposition 3.5. (i) *Assume* $\mathrm{dr}(A) < \infty$*, then there exists* $(F_\lambda, \psi_\lambda, \phi_\lambda)_\Lambda$ n*-decomposable cpc approximations with*
$$A \xrightarrow{\overline{\psi}} \frac{\prod_\Lambda F_\lambda}{\oplus_\Lambda F_\lambda}$$
a **-homomorphism. In particular* A *is strongly quasidiagonal.*

(ii) *Assume* $\dim_{\mathrm{nuc}}(A) < \infty$*, then there exists* $(F_\lambda, \psi_\lambda, \phi_\lambda)_\Lambda$ n*-decomposable cp approximations with*
$$A \xrightarrow{\overline{\psi}} \frac{\prod_\Lambda F_\lambda}{\oplus_\Lambda F_\lambda}$$
order zero.

Both propositions use the following lemma.

Lemma 3.6. *If*
$$A \xrightarrow{\psi} B \xrightarrow{\phi} A$$
cpc and $a \in A_+$ *such that* $\|\phi(\psi(a)) - a\| \leq \eta$, $\|\phi(\psi(a^2)) - a^2\| \leq \eta$*. Then*
$$\|\phi(\psi(a)b) - \phi(\psi(a))\phi(b))\| \leq 3\eta^{1/2}\|b\|, \quad b \in B.$$

This says that if a composition approximates a and a^2 well enough then $\psi(a)$ is approximately in the multiplicative domain of ϕ.

Proof. Stinespring. □

Question 3.7. In view of Theorem 1.7, is there a version of 3.5 for general nuclear C^*-algebras?

Theorem 3.8 (W-Zacharias, Enders)**.** If A is UCT Kirchberg, then $\dim_{\rm nuc}(A) \leq 2$.

Question 3.9. What's the exact value? It can't be zero, as $\dim_{\rm nuc}(A) = 0$ implies A is AF. Does torsion in K-theory imply that $\dim_{\rm nuc}(A) = 2$?

4. The Cuntz semigroup

Recall:

Definition 4.1. $a, b \in A_+$. Say $a \precsim b$, if there is (x_n) in A with $a = \lim x_n^* b x_n$. Write $a \sim b$ if $a \precsim b$ and $b \precsim a$. Define $W(A) = M_\infty(A)_+/\sim$. This is a semigroup. $\mathrm{Cu}(A) = W(A \otimes \mathcal{K})$ (there is a new and subtle story behind this).

Definition 4.2. Suppose A is simple and unital.

(i) A has m-comparison if $d_\tau(a) < d_\tau(b_0), \ldots, d_\tau(b_m)$ for every $\tau \in QT(A)$ implies $[a] \leq [b_0] + \cdots + [b_m]$ in $W(A)$. (Here $a, b_0, \ldots, b_m \in M_\infty(A)_+$).

(ii) $W(A)$ is m-almost unperforated, if

$$(n+1)[a] \leq n[b_0], \ldots, n[b_m] \text{ for some } n \implies [a] \leq [b_0] + \cdots + [b_m].$$

(Note that the $n+1, n$ plays the same role in (ii) that the strict inequality plays on the left hand side of the implication in (i).)

(iii) $W(A)$ is m-almost divisible if whenever $[a] \in W(A)$ and $n \in \mathbb{N}$, then there exists $[b] \in W(A)$ such that $n[b] \leq [a] \leq m(n+1)[b]$

Proposition 4.3 (Rørdam)**.** *If A is simple and unital and $QT(A) = T(A)$ (e.g. if A is exact), then m comparison iff $W(A)$ is m-almost perforated.*

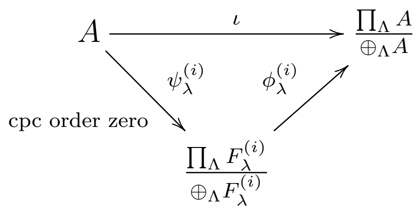
Lemma 4.4 (Kirchberg-Rørdam)**.** *$a, b \in A^1_+$, $\varepsilon > 0$, and $\|a - b\| \leq \varepsilon$, then there exists $d \in A^1$ such that*

$$(a - \varepsilon)_+ = d^* b d$$

Theorem 4.5 (PTWW)**.** $\mathrm{Cu}(\cdot)$ is stable under under small perturbations.

Theorem 4.6 (Robert)**.** $\dim_{\rm nuc}(A) \leq n$ implies A has m-comparison.

Idea of proof. Assume $(k+1)[a] \leq k[b_0]\ldots, b[n_i]$. Then take



The cpc order zero map $\psi^{(i)}$ induces a map at the level of the Cuntz semigroup (this is a key feature of order zero maps). Then get

$$(k+1)[\psi^{(i)}(a)] \leq k[\psi^{(i)}(b_j)]$$

As $\frac{\prod_\Lambda F_\lambda^{(i)}}{\oplus_\Lambda F_\lambda^{(i)}}$ has unperforated Cuntz semigroup, get

$$[\psi^{(i)}(a)] \leq [\psi^{(i)}(b_j)]$$

This gives

$$[\phi^{(i)}\psi^{(i)}(a)] \leq [\phi^{(i)}\psi^{(i)}(a)].$$

Taking sums, gives

$$[\iota(a)] \leq \sum_j [\iota(b_j)]$$

Now adjust to get this back in A. $\square$

5. $\mathcal{Z}$

Definition 5.1. For $p \in \mathbb{N}$, $p \geq 2$ define

$\mathcal{Z}^u_{p,p+1} = C^*(v, s_1, \ldots, s_p :$

(5.1)

$$s_1^* s_1 = s_i s_i^*,\ s_i^* s_i s_j^* s_j = \delta_{i,j}(s_i s_i^*)^2, v^* v = 1 - \sum_k s_k^* s_k,\ vv^* s_1^* s_1 = vv^*)$$

$$= C^*(\Phi, \Psi : \Phi \text{ cpc order 0 on } M_p\ \Psi \text{ cpc order zero on } M_2,$$

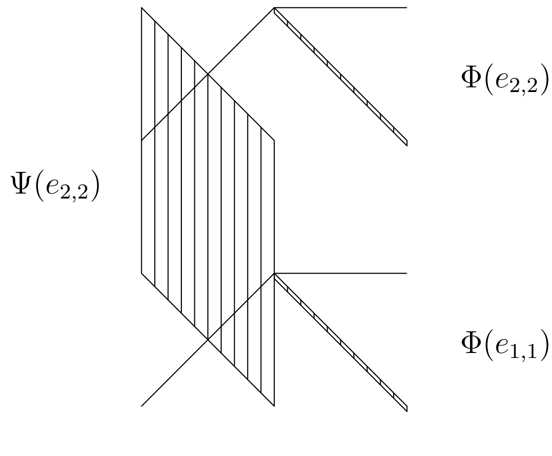
(5.2)

$$\Psi(e_{2,2}) = 1 - \Phi(1_{M_p})),\ \Psi(e_{1,1})\Phi(e_{1,1}) = \Psi(e_{1,1})).$$

(the u stands for universal). Write $\mathcal{R}_{p,p+1}$ for the relations $\Psi(e_{2,2}) = 1 - \Phi(1_{M_p})),\ \Psi(e_{1,1})\Phi(e_{1,1}) = \Psi(e_{1,1})$.

Proposition 5.2 (Rørdam-W)**.** $\mathcal{Z}^u_{p,p+1} \cong \mathcal{Z}_{p,p+1} = \{f \in C([0,1], M_p \otimes M_{p+1}) : f(0) \in M_p \otimes 1_{p+1},\ f(1) \in 1_p \otimes M_{p+1}\}$.

Proof (Sketch), where $p = 2$. We have two positive elements $s_1^* s_1$ and $s_2^* s_2$.



□

With the original description, of commuting cones over M_p and M_{p+1}, it is not easy to show that the generators are semi-projective as commuting is not stable under small perturbations.

Proposition 5.3. *A separable unital is $\mathcal{Z}$-stable if for all (in fact for some) $p \geq 2$, there exist cpc order zero maps $\Phi : M_p \to A_\omega \cap A'$ and $\Psi : M_2 \to A_\omega \cap A'$ such that $\mathcal{R}_{p,p+1}$.*

Proof. Combines 5.2 and 2.10. □

Theorem 5.4 (Rørdam, Jiang)**.** Suppose A separable, simple, unital, stably finite with

$$\overline{\alpha} : A \otimes \mathcal{Z}_{2^\infty,3^\infty} \to A_\omega$$

satisfying

$$\overline{\alpha} \circ (\mathrm{id}_A \otimes 1_{\mathcal{Z}_{2^\infty,3^\infty}}) = \iota_A.$$

Then, A has stable rank one, is K_1-injective, and $W(A)$ is almost unperforated. (the hypothesis is equivalent to $Z_{2^\infty,3^\infty} \hookrightarrow A_\omega \cap A'$).

These things are proved by first doing it for UHF-algebras, and join the arguments for M_{2^∞} and M_{3^∞} along the interval. This usually requires a trick.

We can also write $\mathcal{Z}$ as a universal C^*-algebra:

$$\mathcal{Z}^u = C^*(\Phi_{p_k}^{(k)}, \Psi_2^{(k)} : R_{p_k, o_k+1}, \cdots)$$

the relations which are supressed are quite vicious. Then $\mathcal{Z}^u \cong \mathcal{Z}$, i.e. it is possible to write $\mathcal{Z}$ as a universal C^*-algebra with countably many algebraic generators and relations (Jacelon-W).

Proposition 5.5. *There is a unital trace collapsing* **-homomorphism* $\delta : \mathcal{Z}_{2^\infty,3^\infty} \to \mathcal{Z}_{2^\infty,3^\infty}$

The key ingredient in the proof: Leonel's classification via Cu of morphisms. □

Theorem 5.6 (Rørdam-W)**.** $\mathcal{Z}^\delta := \lim_\to(\mathcal{Z}_{2^\infty,3^\infty}, \delta) \cong \mathcal{Z}$ is separable, simple, unital with unique trace.

Remark 5.7. $\mathcal{Z}^\delta = \mathcal{Z}$ (via Jiang-Su).

Theorem 5.8. $\mathcal{Z}^\delta$ is strongly self absorbing.

Proof. $(\mathcal{Z}^\delta)^{\otimes\infty}$ has stable rank 1, strict comparison by Theorem 5.4. Leonel's result can be used to see that $(\mathcal{Z}^\delta)^{\otimes\infty}$ has approximately inner half flip. Thus $(\mathcal{Z}^\delta)^{\otimes\infty}$ is ssa. By 2.7, we see that $\mathcal{Z}^\delta$ is $(\mathcal{Z}^\delta)^{\otimes\infty}$ stable, so $\mathcal{Z}^\delta \cong (\mathcal{Z}^\delta)^{\otimes\infty}$. □

Theorem 5.9 (Dadarlat-Rørdam, W)**.** If $\mathcal{D}$ is ssa, then $\mathcal{D} \otimes \mathcal{Z}$.

Proof (projection case). Let $p \in \mathcal{D}$ be a non-trivial projection. Define a projection

$$q = p \otimes p + (1-p) \otimes (1-p) \in \mathcal{D} \otimes \mathcal{D}.$$

The projections

$$\begin{array}{rr} p \otimes (1-p) \otimes 1_{\mathcal{D}^{\otimes 2n}} =: e_0 & e_0' := (1-p) \otimes p \otimes 1_{\mathcal{D}^{\otimes 2^n}} \\ \vdots & \vdots \\ q^{\otimes n} \otimes p \otimes (1-p) := e_n & e_n' := q^{\otimes n} \otimes (1-p) \otimes p \end{array}$$

are pairwise orthogonal in $\mathcal{D}^{\otimes 2(n+1)}$. Then $e_0 \oplus \cdots \oplus e_n \sim_{MvN} e_0' \oplus \cdots \oplus e_n'$ (via the approximately inner half flip). Now

$$1 - \left(\sum_{i=0}^n e_i + \sum_{i=0}^n e_i' \right) = q^{\otimes(n+1)} \precsim (1 - q^{\otimes(n-1)}) \otimes 1_{\mathcal{D}^{\otimes 4}}$$

holds for large enough n, using simplicity. This gives the order zero maps Φ, Ψ (in fact *-hms) needed for $\mathcal{Z}$-stability. □

6. $\mathcal{Z}$-stable classification

Definition 6.1. Let $\mathcal{E}$ be a class of separable simple unital nuclear C^*-algebras. Say $\mathcal{E}$ satisfies the (EC) if the following holds: If $A, B \in \mathcal{E}$ and $\Lambda : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$ and isomorphism, there is $\phi : A \stackrel{\cong}{\to} B$ such that $\mathrm{Inv}(\phi) = \Lambda$. Here

$$\mathrm{Inv}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A : T(A) \to S(K_0(A))).$$

Definition 6.2. (i) $\phi : A \otimes \mathcal{Z}_{\underline{p},\underline{q}} \to B \otimes \mathcal{Z}_{\underline{p},\underline{q}}$ (here $\underline{p}, \underline{q}$ are supernatural numbers) a unital $C[0,1]$-morphism. Say ϕ is unitarily suspended, if there is a continuous path $(u_t)_{t\in[0,1)} \subset \mathcal{U}(B \otimes M_{\underline{p}} \otimes M_{\underline{q}})$ such that

$$\phi_t = \mathrm{Ad}(u_t) \circ (\phi_0 \otimes \mathrm{id}_{M_{\underline{q}}}), \quad t \in [0,1).$$

Note that the unitaries could behave very badly near 1, but the *-homomorphism they induce converges (as ϕ_1 exists).

(ii) Say $\Lambda : \mathrm{Inv}(A) \stackrel{\cong}{\to} \mathrm{Inv}(B)$ can be lifted along $\mathcal{Z}_{\underline{p},\underline{q}}$ if there is $\phi : A \otimes \mathcal{Z}_{\underline{p},\underline{q}} \to B \otimes \mathcal{Z}_{\underline{p},\underline{q}}$ a unitarily suspended *-hm $\mathrm{Inv}(\phi_0) = \Lambda_{M_{\underline{p}}}$ and $\mathrm{Inv}(\phi_1) = \Lambda_{M_{\underline{q}}}$. (Here $\Lambda_{M_{\underline{p}}}$ is the isomorphism between $\mathrm{Inv}(A \otimes M_{\underline{p}}) \to \mathrm{Inv}(B \otimes M_{\underline{p}})$ coming from the Kunneth formula.)

Theorem 6.3. Let $\mathcal{E}$ be a class of separable simple unital nuclear C^*-algebras. Suppose that for any $A, B \in \mathcal{E}$, any isomorphism $\Lambda : \mathrm{Inv}(A) \stackrel{\cong}{\to} \mathrm{Inv}(B)$ can be lifted along $\mathcal{Z}_{\underline{p},\underline{q}}$ (for relatively prime supernatural numbers $\underline{p}, \underline{q}$). Then $\mathcal{E}^{\mathcal{Z}} := \{A \otimes \mathcal{Z} : A \in \mathcal{E}\}$ satisfies (EC).

The proof is an intertwining argument.

Theorem 6.4 (Lin,Lin-Niu,Lin-W,...)**.** For

$$\mathcal{E} := \{A \text{ separable, simple, unital, nuclear, UCT such that } A \otimes \mathcal{Q} \text{ is } TAI\},$$

the class $\mathcal{E}^{\mathcal{Z}}$ satisfies (EC). (Here $\mathcal{Q}$ is the universal UHF).

The crucial difficulty of the proof. Given $\Lambda : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$, can lift $\Lambda_{M_{\underline{p}}}$ and $\Lambda_{M_{\underline{q}}}$ to ϕ_0, ϕ_1 respectively. Then, $\phi_0 \otimes \mathrm{id}_{M_{\underline{q}}} \approx_{au} \phi_1 \otimes \mathrm{id}_{M_{\underline{p}}}$. We want assymptotically unitarily equivalence $\phi_0 \otimes \mathrm{id}_{M_{\underline{q}}} \approx_{asu} \phi_1 \otimes \mathrm{id}_{M_{\underline{p}}}$ (i.e. a continuous path of unitaries doing this). This is not always true: there's an obstruction. Lin-Niu modify ϕ_0 and ϕ_1 to remove the obstruction. □

TASKS:

- Confirm TAI (after tensoring with $\mathcal{Q}$)
- Generalise TAI classification.

Strategy for classification.

(i) Classify $A \otimes \mathcal{Z}$ (Section 6).
(ii) Show that $A \cong A \otimes \mathcal{Z}$ (Section 7).

This has been done before. When N is a II_1 factor

(i) McDuff and injective factors are R
(ii) injective $\implies$ McDuff.

The question of when A is $\mathcal{Z}$-stable gets at the very heart of the subject.

7. The regularity conjecture

Conjecture 7.1 (Toms-W)**.** Let A be separable simple unital, non elementary, nuclear. TFAE:

(i) $\dim_{\mathrm{nuc}}(A) < \infty$;
(ii) A is $\mathcal{Z}$-stable;
(iii) A has strict comparison (0-comparison).

Remark 7.2.
- It's tempting to add: (iii') A has strict comparison and almost divisibility.
- Also we might add: (iv) A has almost divisibility
- and: (v) Sato's property (SI) a version of strict comparison inside the central sequence algebra.
- If A is stably finite, modify the conjecture by replacing $\dim_{\mathrm{nuc}}(A)$ by $\mathrm{dr}(A)$.
- None of these implications is trivial (except those that obviously are: (iii') implies (iii) and (iii') implies (iv)).

Theorem 7.3. We have

- $(i) \implies (ii)$ [W] [in fact $(i) \implies (m,m')$-(iii')] (i.e. m comparison and m' almost divisibility for some m, m' depending on $\dim_{\mathrm{nuc}}(A)$ and this is enough to get $\mathcal{Z}$ stability.
- $(ii) \implies (iii') \implies (iii), (iv)$ [Rørdam].
- $(ii) \implies (v)$ [Matui-Sato]
- $(iii') \implies (ii)$ if A has locally finite nuclear dimension [W]. [6]
- $(iii) \implies (iii') \implies (iv)$ if $\partial_e(T(A))$ is compact and finite dimensional (Dadarlat-Toms).
- $(iii) \implies (v)$ if $\partial_e(T(A))$ is finite [Matui-Sato]
- $(v) \implies (ii)$ if $\partial_e(T(A))$ is finite [Matui-Sato].

[6] The implication $(iii') \implies (ii)$ under the hypothesis of locally finite nuclear dimension, can be viewed in the following way: $A \cong A \otimes \mathcal{Z} \Leftrightarrow \mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{Z})$ when A has locally finite nuclear dimension. Thus we can detect $\mathcal{Z}$-stability using the Cuntz semigroup.

- $(ii) \implies (i)$ if:
  - if we have classification and finite dimensional models (for example Kirchberg-Philips classification).
  - Gong's reduction theorem
  - if A is locally homogeneous [Tikuisis-W] (this even holds in the non-simple case).

Remark 7.4. $(iii) \implies (v) \implies (ii)$ (for $\partial_e(T(A))$ finite) is a unified version of Kirchberg's $\mathcal{O}_\infty$ absorption theorem, i.e. it unifies the finite and the infinite case.

Problem 7.5. Does $(iii') \implies (iv) + (v) \implies (ii)$ with Matui-Sato's methods (but without assuming finitely many extremal traces)?

Theorem 7.6 (Tikuisis-W)**.** X compact metrisable. Then $\mathrm{dr}(C(X) \otimes \mathcal{Z}) \leq 2$.

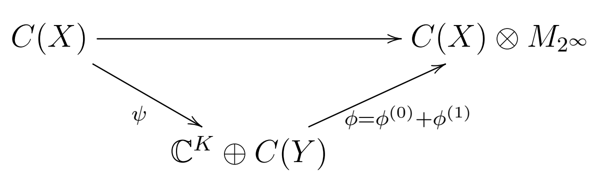
Idea of proof. We'll restrict to $C(X) \otimes M_{2^\infty}$, then some extra manipulations give the result. By Voiculescu $C(0,1] \otimes \mathcal{O}_2$ is quasidiagonal. Thus

$$C(0,1] \otimes \mathcal{O}_2 \hookrightarrow (M_{2^\infty})_\omega$$

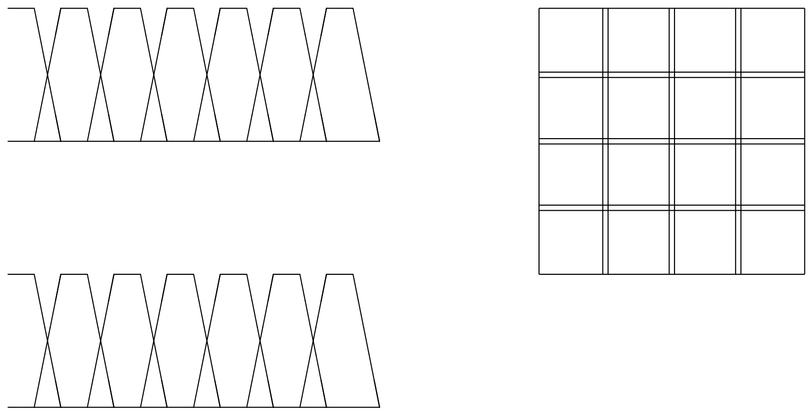
and the image will be tracially small (in the kernel of all traces on the ultraproduct). To control the decomposition rank of $C(X) \otimes M_{2^\infty}$ we need to control the decomposition rank of the embedding

$$C(X) \to C(X) \otimes M_{2^\infty}.$$

To do this find,



with $Y \subset X$ closed but potentially high dimensional and $\phi^{(0)}, \phi^{(1)}$ cpc order zero and such that $\phi^{(1)}$ has tracially small image. This is done by means of the following mysterious picture.



We get

$$\begin{array}{ccccc} C_0((0,1]\times Y) & \longrightarrow & C_0(0,1]\otimes C(Y)\otimes \mathcal{O}_2 & \longrightarrow & (C(X)\otimes M_{2^\infty})_\omega \\ & \underset{\text{Kirchberg-Rørdam}}{\searrow} & & \nearrow & \\ & & C_0(\Gamma)\otimes\mathcal{O}_2 & & \end{array}$$

Γ graph

Kirchberg-Rørdam's argument uses $\mathcal{U}(C(S^1,\mathcal{O}_2))$ connected. Now see that $\phi^{(1)}$ uses 2-colours and $\phi^{(0)}$ uses 1 colours, leading us to 3 colours and dimension at most 2. □

8. Minimal dynamical systems

Notation 8.1. Let X be compact metrisable, $T : X \curvearrowright$ minimal homeomorphism and $\alpha : C(X) \curvearrowright$ automorophism induced by T, $\alpha(f)(x) = f\circ T^{-1}(x)$. Define $\mathcal{A} = C(X)\rtimes_\alpha \mathbb{Z} = C^*(C(X), u : \alpha(f) = ufu^*)$.

Definition 8.2. For $Y\subset X$ closed, then $\mathcal{A}_Y := C^*(C(X), uX_0(X\setminus Y))$. (Write $\mathcal{A}_y$ rather than $\mathcal{A}_{\{y\}}$ when $Y = \{y\}$).

Proposition 8.3 (Putnam, Lin-Philips)**.**

- *If $Y = \bigcap_n Y_n$ a decreasing sequence, then $\mathcal{A}_Y = \lim_{n\to\infty}\mathcal{A}_{Y_n}$.*
- *If $Y = \{y\}$, then $\mathcal{A}_y$ is simple.*
- *$K_0(\mathcal{A}_y)\cong K_0(\mathcal{A})$ and $T(\mathcal{A}_y)\cong T(\mathcal{A})$*
- *Y non-empty interior then $\mathcal{A}_y$ RSH (with canonical decomposition as an iterated pull back).*

Theorem 8.4 (Lin-Philips, Strung-W)**.** $\mathcal{A}_y \otimes \text{UHF}$ is TAF/I $\Longrightarrow \mathcal{A} \otimes$ UHF is TAF/I.[7]

Proof. Uses Berg's technique. □

Theorem 8.5 (Toms-W)**.** If $\dim(X) < \infty$, then $\dim_{\text{nuc}}(C(X) \rtimes_\alpha \mathbb{Z}) < \infty$ and $C(X) \rtimes \mathbb{Z}$ is $\mathcal{Z}$-stable.

The crucial point is to consider x, y in different orbits and then

$$\begin{array}{ccc} \mathcal{A} & & \mathcal{A} \\ & \searrow \quad \nearrow_{\phi \text{ sum of 2 }^*\text{-hms}} & \\ & \mathcal{A}_x \oplus \mathcal{A}_y & \end{array}$$

We find approximate maps from $\mathcal{A}$ to $\mathcal{A}_x$ and $\mathcal{A}_y$. Can get $\dim_{\text{nuc}}(\mathcal{A}_x), \dim_{\text{nuc}}(\mathcal{A}_y) \leq \dim(X)$ by the RSH-decomposition.

Theorem 8.6.

$$\mathcal{E} = \{C(X) \rtimes_\alpha \mathbb{Z} : X \text{ infinite, compact, metrisable, } \dim(X) < \infty, \\ \alpha \text{ induced by uniquely ergodic minimal homeo}\}$$

satisfies (EC).

Proof. $\mathcal{A} \in \mathcal{E}$ implie $\mathcal{A}_y \otimes \text{UHF}$ simple, monotracial, real rank zero, ASH, hence TAF. By Theorem 8.4, $\mathcal{A} \otimes \text{UHF}$ is TAF. By Theorem 6.4, $\mathcal{E}^{\mathcal{Z}}$ satisfies (EC). By Theorem 8.5 $\mathcal{E}^{\mathcal{Z}} = \mathcal{E}$. □

This works if projections separate traces.

Problem 8.7. What if projections do not separate traces?

9. Dynamics and dimension

(X, T) as before.

Definition 9.1. (i) (Hirschberg, W, Zacharias) $\dim_{\text{Rok}}(X, T) \leq n$ if for every $L \in \mathbb{N}$, there are open subsets $U_l^{(m)} \subset X$, $m \in \{0, \cdots, n\}$, $l \in \{1, \cdots, L\}$ satisfying:
 (a) $T^{-1}(U_l^{(m)}) = U_{l+1}^{(m)}$ for $m \in \{0, \cdots, n\}$ and $l \in \{1, \cdots, L-1\}$.
 (b) $U_l^{(m)} \cap U_{l'}^{(m)} = \emptyset$ for $l \neq l'$
 (c) $(U_l^{(m)})_{m,l}$ covers X.
This definition can be satisfied for one factor without being satistied for the whole system: hence the next definition.

[7] Lin-Phillips proved that $\mathcal{A}_y$ is TAF implies that $\mathcal{A}$ is TAF when $\mathcal{A}$ is real rank zero, and the argument is to tensor the proof by a UHF.

(ii) $\dim(X,T) \leq n$ if for every $L \in \mathbb{N}$ if for every open cover $\mathcal{V}$ of X there are open $U_{k,l}^{(m)} \subset X$ where $m \in \{0,1,\cdots,n\}$, $k = \{1,\cdots,K^{(m)}\}$ and $l \in \{1,\cdots,L\}$ satisfying:
(a) $T^{-1}(U_{k,l}^{(m)}) = U_{k,l+1}^{(m)}$ or $m \in \{0,\cdots,n\}$, $k \in \{1,\cdots,K^{(m)}\}$ and $l \in \{1,\cdots,L-1\}$.
(b) $U_{k,l}^{(m)} \cap U_{k',l'}^{(m)} = \emptyset$ if $(k,l) \neq (k',l')$.
(c) $(U_{m,k}^{(m)})_{m,k,l}$ is an open cover
(d) $(T^{-p}U_{k,l}^{(m)})$ refines $\mathcal{V}$ for each $p \in \{0,\cdots,L\}$.
Having $\dim(X,T) \leq n$ implies $\dim(X) \leq n$, hence the reason for the next definition.

(iii) (X,T) has *slow dynamic dimension growth* if: for every $\varepsilon > 0$ and every open cover $\mathcal{V}$ of X, there are $n, L, K^{(0)}, \cdots, K^{(m)}$ and open subsets $(U_{k,l}^{(m)})_{m,k,l}$ satisfying (a), (b), (c), (d) above and $(n+1)/L < \varepsilon$.

Remarks 9.2. (i) Interpret n as the number of colours, L as the length of the Roklin towers.
(ii) If a factor has finite Roklin dimension, then so too does the whole system.

Proposition 9.3. $\dim(X,T) < \infty \Longleftrightarrow \dim_{\mathrm{Rok}}(X,T) < \infty$ *and* $\dim(X) < \infty$.

Implication from left to right is trivial, from right to left provides an easier way of getting at the dynamic dimension.

Recall:

$$\operatorname{m}\dim(X,T) = \sup_{\mathcal{V}} \lim_{L\to\infty} \frac{1}{L} D(\mathcal{V} \vee T\mathcal{V} \vee \cdots \vee T^L\mathcal{V})$$

where $D(\cdot)$ is the minimum order of a refinement.

Theorem 9.4. (X,T) has slow dynamic dimension growth $\Longrightarrow$ $\operatorname{m}\dim(X,T) = 0$. Conversely, $\operatorname{m}\dim(X,T) = 0$ and $\dim_{\mathrm{Rok}}(X,T) < \infty \Longrightarrow (X,T)$ has slow dynamic dimension growth.

Condition (d) in slow dynamic dimension growth (or dynamic dimension) ensures that $(U_{k,l}^{(m)})_{m,k,l}$ refines $\mathcal{V} \vee T\mathcal{V} \vee \cdots \vee T^L\mathcal{V}$.

Theorem 9.5 (Hirschberg, W, Zacharias)**.** If (X,T) is minimal and $\dim(X) < \infty$, then $\dim_{\mathrm{Rok}}(X,T) < \infty$ and so $\dim(X,T) < \infty$.

Proof. Heavily uses RSH structure of $\mathcal{A}_y$. □

Definition 9.6. (i) Given U, V open subsets of X. Define $U \prec_m V$ if the following holds. For any compact $Y \subset U$, there are $(U_k^{(i)})$

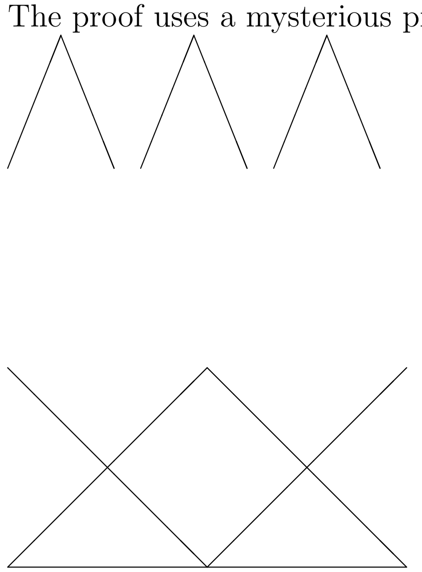
open subsets of U for $i \in \{0, \dots, m\}$ and $k \in \{1, \dots, K^{(i)}\}$ and $(V_k^{(i)})$ open subsets of V for $i \in \{0, \dots, m\}$ and $k \in \{1, \dots, K^{(i)}\}$ of V such that:

- for each i, k there is $r_k^{(i)} \in \mathbb{Z}$ such that $T^{r_k^{(i)}}(U_k^{(i)}) \subset V_k^{(i)}$ (i.e. each $U_k^{(i)}$ transported under the $V_k^{(i)}$ but it's not prescribed how long it should take).
- for each i, $V_k^{(i)} \cap V_{k'}^{(i)} = \emptyset$ for $k \neq k'$.
- $(U_k^{(i)})_{i,k}$ cover Y.

(ii) (X, T) has m-comparison if whenever $U, V \in X$ are open such that with $\mu(U) < \mu(V)$ for all regular invariant Borel measures μ, then $U \prec_m V$.

Theorem 9.7. $\dim(X, T) \leq m \implies (X, T)$ has m-comparison.

The proof uses a mysterious picture.



Definition 9.8. A dynamic version of $\mathcal{Z}$-stability. The definition will be provided once it satisfies the following theorem.

Theorem to be 9.9.

$$(X, T) \text{ is dynamically } \mathcal{Z}\text{-stable} \implies \begin{cases} \dim_{\mathrm{Rok}}(X, T) \leq 1 \\ C(X) \rtimes_\alpha \mathbb{Z} \quad \mathcal{Z}\text{-stable} \\ \text{dynamic comparison} \end{cases}$$

Question 9.10. $C(X) \rtimes \mathbb{Z}$ $\mathcal{Z}$-stable $\implies (X, T)$ dynamically $\mathcal{Z}$-stable? $\dim_{\mathrm{nuc}}(C(X) \rtimes \mathbb{Z})) < \infty \implies \dim(X, T) < \infty$? and also for comparison.

10. Outlook and open problems

Problems:

(1) A is finite and $\dim_{\mathrm{nuc}}(A) < \infty \implies \mathrm{dr}(A) < \infty$?
(2) A nuclear $\implies$ A has locally finite nuclear dimension?
(3) non-simple and nonunital version of the the regularity conjecture.
(4) Range result for $\mathrm{Cu}(\cdot)$ for simple nuclear C^*-algebras.
(5) Unifed classification results (purely (in)finite).
(6) TAS classiciation (S is a suitable class of 1-dimensional subhomogeneous algebras, such as splitting interval algebras).
(7) Connes's odd spheres (classify the crossed products in the non-uniquely ergodic situation).
(8) Free minimal $\mathbb{Z}^d$ actions? Need replacement for $\mathcal{A}_y$.
(9) $\mathcal{Z}$ as a crossed product?
(10) dynamical version of $\mathrm{Cu}(\cdot)$?
(11) is there a dynamical proof that $\dim(X,T) < \infty$ when $\dim(X) < \infty$?
(12) are there appplications of $\dim(X,T) < \infty$ within dynamical systems?
(13) Interpret GPS as a zero dimensional incarnation of higher dimensional phenomena (not restricting to Cantor spaces any more)