Tauberian Theorems in Analysis and Number Theory

DIMITAR K. DIMITROV
State University of São Paulo, Brazil
Email: dimitar.k.dimitrov@unesp.br

We formulate briefly two of the most challenging questions in Number Theory: the Gauss circle problem and the one for the second term in the asymptotics in the Prime Number Theorm, the latter being essentially the Riemann hypothesis. Then we pose a general question about possible refinements of the classical Tauberian theorems. Given a sequence of real numbers, it requires conditions on either the corresponding Dirichelet series or the discrete Laplace transform that would eventually provide information about the second term in the asysmptotic behavior of the partial sums of the series, formed by the given sequence.

Then we discuss a result which provides a partial answer to a very old open problem of Pólya concerning necessary and sufficient conditions on a kernel in order that its Fourier transform is an entire function with only real zeros. This is a joint work with Yuan Xu, the proof uses one of Wiener's Tauberian theorems, so that it is a density criterion, in terms of translates of the kernel, in $L^1(\mathbb{R})$. The result yields another necessary and sufficient condition, probably another totally useless one, for the Riemann hypothesis. From this point of view the result may be seen as an analog of the celebrated Nyman-Beurling and Báez-Duarte's necessary and sufficient conditions, which are also a density criteria, but in $L^2[0, 1]$ and $L^2(\mathbb{R})$.