Powers of Matrix Differential Operators and Applications

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In their study of the asymptotic iteration method (AIM), Ismail and Saad [J. Math. Phys. **61**, 033501 (2020)] posed the problem of finding an explicit formula for powers of a matrix differential operator associated with the AIM sequences. We address this problem by placing it in the more general setting of matrix differential operators of the form

$$\mathcal{L} = D + M(x), \qquad D = \frac{d}{dx},$$

where $M(x)$ is an arbitrary square matrix whose entries are functions of x. An explicit expansion for $\mathcal{L}^n$ is obtained in terms of recursively generated coefficient matrices. A complementary representation is also developed in terms of an invertible matrix satisfying a first-order matrix differential equation. These representations also allow certain operational formulas of Al-Salam and Ismail [J. Math. Anal. Appl. **51** (1975), 208–218] to be applied naturally in the matrix-differential setting.

For the matrix arising from AIM, the coefficient matrices are identified explicitly with the AIM sequences. This yields the desired operator-power formula together with finite-sum representations for the AIM sequences and the associated termination quantity. In these representations, differentiation acts only on the original coefficient functions, thereby avoiding the repeated differentiation inherent in the classical AIM recurrence.

Several consequences are discussed. For second-order linear differential equations, whenever a polynomial solution exists, the first stabilized AIM ratio determines the polynomial up to normalization and coincides with its negative logarithmic derivative; consequently, the zeros of the polynomial are encoded as the poles of this rational function. We also indicate how the same framework leads naturally to finite-term recurrences for the AIM sequences associated with classical hypergeometric-type equations and Heun-type equations.