Analysis and Applications, Vol. 10, No. 2 (2012) 215–235

DOI: 10.1142/S0219530512500108

ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS VIA RECURRENCE RELATIONS

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Received 19 January 2011
Accepted 20 January 2011
Published 23 March 2012

We use the Legendre polynomials and the Hermite polynomials as two examples to illustrate a simple and systematic technique on deriving asymptotic formulas for orthogonal polynomials via recurrence relations. Another application of this technique is to provide a solution to a problem recently raised by M. E. H. Ismail.

Keywords: Asymptotics; orthogonal polynomials; recurrence relation; Legendre polynomials; Hermite polynomials.

Mathematics Subject Classification 2010: 41A60, 33C45

1. Introduction

There are many powerful and systematically developed techniques in asymptotic theory for orthogonal polynomials. For instance, the steepest-descent method for integrals [9], the WKB (Liouville–Green) approximation for differential equations [7], the Deift–Zhou's nonlinear steepest-descent method for Riemann-Hilbert problems [2, 3], etc. Here, we intend to develop a simple, and yet systematic approach to derive asymptotic formulas for orthogonal polynomials by using their recurrence relations. Let $\{\pi_n(x)\}_{n=0}^{\infty}$ be a system of monic polynomials satisfying the recurrence relation

$$\pi_{n+1}(x) = (x - a_n)\pi_n(n) - b_n\pi_{n-1}(x), \quad n \geq 1, \tag{1.1}$$

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and the initial conditions $\pi_0(x) = 1$ and $\pi_1(x) = x - a_0$. Note that for the sake of convenience, we have normalized the polynomials to be monic. To construct the asymptotic formulas of $\pi_n(x)$, we first set

$$\pi_n(x) = \prod_{k=1}^{n} w_k(x). \tag{1.2}$$

It is readily seen from (1.1) that $w_1(x) = x - a_0$ and

$$w_{k+1}(x) = x - a_k - \frac{b_k}{w_k}, \quad k \geq 1. \tag{1.3}$$

When x is away from the oscillatory region of the orthogonal polynomials, it is easy to find an asymptotic formula for $w_k(x)$ from (1.3). Then, as we shall see, the asymptotic behavior of $\pi_n(x)$ for x away from the oscillatory region can be obtained readily. When x is near the oscillatory region, we use a method similar to that given in [8, 10] to derive asymptotic formulas for general solutions of (1.1). The asymptotic formula of $\pi_n(x)$ for x near the oscillatory region is then obtained by doing a matching. In the subsequent three sections, we will consider the following three cases:

Case 1: $a_n = 0$ and $b_n = n^2/(4n^2 - 1)$. This case is related to the Legendre polynomials.
Case 2: $a_n = 0$ and $b_n = n/2$. This case is related to the Hermite polynomials.
Case 3: $a_n = n^2$ and $b_n = 1/4$. This case was recently brought to our attention by Ismail.

For simplicity, we use the same notations in the following three sections. Since each section is independent and self-contained, this will not lead to any confusion.

2. Case 1: The Legendre Polynomials

The Legendre polynomials can be defined as [6, (1.8.57)]

$$P_n(x) = {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix} \,\middle|\, \frac{1-x}{2} \right).$$

They satisfy the recurrence relation [6, (1.8.59)]

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

For convenience, we normalize the Legendre polynomials to be monic. Put

$$\pi_n(x) := \frac{2^n n!}{(n+1)_n} P_n(x).$$

The monic Legendre polynomials $\{\pi_n(x)\}_{n=0}^{\infty}$ satisfy [6, (1.8.60)]

$$\pi_{n+1}(x) = x\pi_n(x) - \frac{n^2}{4n^2-1}\pi_{n-1}(x), \quad n \geq 1, \tag{2.1}$$

$$\pi_0(x) = 1, \quad \pi_1(x) = x. \tag{2.2}$$

Theorem 2.1. *As $n \to \infty$, we have*

$$\pi_n(x) \sim \left(\frac{x+\sqrt{x^2-1}}{2}\right)^n \left(\frac{x+\sqrt{x^2-1}}{2\sqrt{x^2-1}}\right)^{1/2} \tag{2.3}$$

for x in the complex plane bounded away from $[-1,1]$.

Proof. Set

$$\pi_n(x) = \prod_{k=1}^{n} w_k(x). \tag{2.4}$$

From (2.1), (2.2) and (2.4), it follows that

$$w_{k+1}(x) = x - \frac{k^2}{4k^2-1}\frac{1}{w_k(x)}, \quad k \geq 1, \tag{2.5}$$

$$w_1(x) = x. \tag{2.6}$$

As $k \to \infty$, we have

$$w_k(x) \sim \frac{x+\sqrt{x^2-1}}{2}$$

for $x \in \mathbb{C}\backslash[-1,1]$. Here, the square root takes its principle value so that $\sqrt{x^2-1} \sim x$ as $x \to \infty$. Define

$$w(x) := \frac{x+\sqrt{x^2-1}}{2} \tag{2.7}$$

and

$$u_k(x) := \frac{w_k(x)}{w(x)}. \tag{2.8}$$

It is easily seen from (2.5), (2.6) and (2.8) that

$$u_{k+1}(x) = \frac{x}{w(x)} - \frac{k^2}{4k^2-1}\frac{1}{w(x)^2 u_k(x)}, \quad k \geq 1, \tag{2.9}$$

$$u_1(x) = \frac{x}{w(x)}. \tag{2.10}$$

We make a change of variable

$$t = t(x) := (x - \sqrt{x^2 - 1})^2. \tag{2.11}$$

It follows from (2.7) and (2.11) that

$$w(x)^2 = \frac{1}{4t}, \qquad \frac{x}{w(x)} = 1 + t. \tag{2.12}$$

Hence, Eqs. (2.9) and (2.10) can be written as

$$u_{k+1}(x) = 1 + t - \frac{4k^2 t}{4k^2 - 1}\frac{1}{u_k(x)}, \qquad k \geq 1, \tag{2.13}$$

$$u_1(x) = 1 + t. \tag{2.14}$$

Define $Q_0(t) := 1$ and

$$Q_n(t) := \prod_{k=1}^{n} u_k(x), \qquad n \geq 1. \tag{2.15}$$

From (2.13)–(2.15), we obtain $Q_1(t) = 1 + t$ and

$$Q_{n+1}(t) = (1 + t)Q_n(t) - \frac{4n^2 t}{4n^2 - 1}Q_{n-1}(t).$$

From this recurrence relation, one can construct a generating function from which it is easily deducible that $Q_n(t)$ has the explicit expression

$$Q_n(t) = \sum_{j=0}^{n} \frac{(1/2)_j (n - j + 1)_j}{j!(n - j + 1/2)_j} t^j.$$

A simpler verification of this identity is by induction. Using the Lebesgue dominated convergence theorem, it can be readily shown that

$$Q_n(t) \to (1 - t)^{-1/2}$$

as $n \to \infty$. Note that by (2.4), (2.8) and (2.15), $\pi_n(x) = w(x)^n Q_n(t)$. Thus, it follows that

$$\pi_n(x) \sim w(x)^n (1 - t)^{-1/2}$$

as $n \to \infty$. This, together with (2.7) and (2.11), yields (2.3). □

Theorem 2.2. *Let $\delta > 0$ be any fixed small number. For x in a small complex neighborhood of the interval $[-1 + \delta, 1 - \delta]$, we have*

$$\pi_n(x) \sim \frac{1}{2^n}\left[\cos n\theta \left(\frac{1 + \sin\theta}{\sin\theta}\right)^{1/2} + \sin n\theta \left(\frac{1 - \sin\theta}{\sin\theta}\right)^{1/2}\right] \tag{2.16}$$

as $n \to \infty$, where $\theta = \theta(x) := \arccos x$ with $0 < \operatorname{Re}\theta < \pi$.

Proof. To put the difference Eq. (2.1) in the form suggested by Wang and Wong [8, (2.1)], we let

$$p_n(x) := \frac{2^n\Gamma(n/2+1/4)\Gamma(n/2+3/4)}{[\Gamma(n/2+1/2)]^2}\pi_n(x). \tag{2.17}$$

From (2.17), it is easily seen that

$$\frac{[\Gamma(n/2+1/2)]^2(n/2+1/4)}{[\Gamma(n/2+1)]^2}\cdot 2xp_n(x) = p_{n+1}(x) + p_{n-1}(x). \tag{2.18}$$

Motivated by the form of the normal (series) solutions to second-order difference equations (see [10, (1.5)]), we assume

$$p_n(x) \sim n^\alpha[r(x)]^n\{f(x)\cos[n\varphi(x)] + g(x)\sin[n\varphi(x)]\} \tag{2.19}$$

as $n\to\infty$, where $r(x)$ and $\varphi(x)$ are real-valued functions, whereas $f(x)$ and $g(x)$ can be complex-valued. We now proceed to determine the constant α and the functions $r(x)$, $\varphi(x)$, $f(x)$ and $g(x)$ in (2.19). It can be easily shown from (2.19) that

$$p_{n\pm1}(x) \sim n^\alpha r^{n\pm1}[(f\cos\varphi \pm g\sin\varphi)\cos(n\varphi) + (g\cos\varphi \mp f\sin\varphi)\sin(n\varphi)]. \tag{2.20}$$

Furthermore, by the asymptotic formula for the ratio of Gamma functions [1, (6.1.47)], we have

$$\frac{[\Gamma(n/2+1/2)]^2(n/2+1/4)}{[\Gamma(n/2+1)]^2} = 1 + O(n^{-2}) \tag{2.21}$$

as $n\to\infty$. Applying (2.19)–(2.21) to (2.18) gives

$$\begin{aligned} &2x[f\cos(n\varphi) + g\sin(n\varphi)] \\ &\quad\sim r[(f\cos\varphi + g\sin\varphi)\cos(n\varphi) + (g\cos\varphi - f\sin\varphi)\sin(n\varphi)] \\ &\qquad + r^{-1}[(f\cos\varphi - g\sin\varphi)\cos(n\varphi) + (g\cos\varphi + f\sin\varphi)\sin(n\varphi)]. \end{aligned}$$

Comparing the coefficients of $\cos(n\varphi)$ and $\sin(n\varphi)$ on both sides of the last formula yields

$$\begin{aligned} 2xf &= (r+r^{-1})f\cos\varphi + (r-r^{-1})g\sin\varphi; \\ 2xg &= -(r-r^{-1})f\sin\varphi + (r+r^{-1})g\cos\varphi. \end{aligned}$$

Thus, we obtain from the above two equations

$$x = \cosh(\log r)\cos\varphi, \quad 0 = \sinh(\log r)\sin\varphi.$$

It can be easily seen that the only solution to these equations is $\log r = 0$ and $\varphi = \arccos x$. Recall that

$$\theta = \theta(x) := \arccos x, \quad 0 < \operatorname{Re}\theta < \pi. \tag{2.22}$$

Hence, we conclude that

$$r = 1 \quad \text{and} \quad \varphi = \theta. \tag{2.23}$$

Next, we are going to determine the constant α in (2.19). Applying (2.23) to (2.19) gives

$$p_n(x) \sim n^\alpha[f\cos(n\theta) + g\sin(n\theta)], \tag{2.24}$$

and

$$p_{n\pm1}(x) \sim n^\alpha\left(1 \pm \frac{\alpha}{n}\right)[(f\cos\theta \pm g\sin\theta)\cos(n\theta) + (g\cos\theta \mp f\sin\theta)\sin(n\theta)]. \tag{2.25}$$

A combination of (2.18), (2.21), (2.24) and (2.25) yields

$$\begin{aligned} 2x[f\cos(n\theta) + g\sin(n\theta)] &\sim 2f\cos\theta\cos(n\theta) + 2g\cos\theta\sin(n\theta) \\ &\quad + \frac{\alpha}{n}[2g\sin\theta\cos(n\theta) - 2f\sin\theta\sin(n\theta)]. \end{aligned}$$

In view of (2.22), we obtain by matching the coefficients in the last formula

$$\alpha g\sin\theta = 0, \quad \alpha f\sin\theta = 0.$$

These equations hold for all x in a small complex neighborhood of $[-1+\delta, 1-\delta]$. Since f and g cannot be identically zero, it follows that

$$\alpha = 0. \tag{2.26}$$

Thus, we have from (2.19), (2.23) and (2.26)

$$p_n(x) \sim f\cos n\theta + g\sin n\theta \tag{2.27}$$

as $n \to \infty$. This formula holds uniformly for x in a small complex neighborhood of $[-1+\delta, 1-\delta]$. Moreover, it follows from (2.3) and (2.17) that

$$p_n(x) \sim (x + \sqrt{x^2-1})^n \left(\frac{x + \sqrt{x^2-1}}{2\sqrt{x^2-1}}\right)^{1/2} \tag{2.28}$$

for complex x bounded away from $[-1, 1]$. Our last step is to determine the coefficients f and g in (2.27) by matching the above two formulas in an overlapping region. With θ and x given by (2.22), it can be shown that for $\operatorname{Im} x > 0$, we have $\operatorname{Im}\theta < 0$. Thus, (2.27) implies

$$p_n(x) \sim \left(\frac{f}{2} + \frac{g}{2i}\right) e^{in\theta}.$$

Meanwhile, in view of $x = \cos\theta$ and $\sqrt{x^2-1} = i\sin\theta$ by (2.22), we obtain from (2.28) that

$$p_n(x) \sim e^{in\theta} \left[\frac{e^{i(\theta-\pi/2)}}{2\sin\theta}\right]^{1/2}.$$

Coupling the last two formulas gives

$$\frac{f}{2} + \frac{g}{2i} = \frac{e^{i(\theta/2-\pi/4)}}{(2\sin\theta)^{1/2}}.$$

Similarly, matching (2.27) with (2.28) in the region $\operatorname{Im} x < 0$ yields

$$\frac{f}{2} - \frac{g}{2i} = \frac{e^{-i(\theta/2-\pi/4)}}{(2\sin\theta)^{1/2}}.$$

From the last two equations of f and g we have

$$f = \left(\frac{1+\sin\theta}{\sin\theta}\right)^{1/2}, \quad g = \left(\frac{1-\sin\theta}{\sin\theta}\right)^{1/2}.$$

This, together with (2.17) and (2.27), implies (2.16). □

3. Case 2: The Hermite Polynomials

The Hermite polynomials can be defined as [6, (1.13.1)]

$$H_n(x) = (2x)^n {}_2F_0\left(\begin{matrix} -n/2, & -(n-1)/2 \\ & - \end{matrix} \middle| -\frac{1}{x^2}\right).$$

They satisfy the recurrence relation [6, (1.13.3)]

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x).$$

For convenience, we normalize the Hermite polynomials to be monic, and put

$$\pi_n(x) := 2^{-n}H_n(x).$$

The monic Hermite polynomials $\{\pi_n(x)\}_{n=0}^{\infty}$ satisfy [6, (1.13.4)]

$$\pi_{n+1}(x) = x\pi_n(x) - \frac{n}{2}\pi_{n-1}(x), \quad n \geq 1, \tag{3.1}$$

$$\pi_0(x) = 1, \quad \pi_1(x) = x. \tag{3.2}$$

Theorem 3.1. *As $n \to \infty$, we have*

$$\pi_n(\sqrt{2n}y) \sim \left(\frac{n}{2e}\right)^{n/2} \exp\left\{n[y^2 - y\sqrt{y^2-1} + \log(y+\sqrt{y^2-1})]\right\} \times \left(\frac{y+\sqrt{y^2-1}}{2\sqrt{y^2-1}}\right)^{1/2} \tag{3.3}$$

for complex y bounded away from the interval $[-1, 1]$.

Proof. Set

$$\pi_n(x) = \prod_{k=1}^{n} w_k(x). \tag{3.4}$$

It follows from (3.1) and (3.2) that $w_1(x) = x$ and

$$w_{k+1}(x) = x - \frac{k}{2w_k(x)}.$$

Let $x = x_n := \sqrt{2n}y$ with $y \in \mathbb{C}\backslash[-1, 1]$. It can be proved by induction that for real y and $y \notin [-1, 1]$, we have

$$\frac{x_n + \sqrt{x_n^2 - 2k}}{2}\left[1 + \frac{1}{2(x_n^2-2k)} - \frac{5x_n - \sqrt{x_n^2-2k}}{8(x_n^2-2k)^{5/2}}\right] < w_k(x_n) < \frac{x_n + \sqrt{x_n^2-2k}}{2}\left[1 + \frac{1}{2(x_n^2-2k)}\right]$$

for all $k = 1, \ldots, n$. From these inequalities, it follows that

$$w_k(x_n) = \frac{x_n + \sqrt{x_n^2-2k}}{2}\left[1 + \frac{1}{2(x_n^2-2k)} + O(n^{-2})\right] \tag{3.5}$$

as $n \to \infty$, uniformly in $k = 1, \ldots, n$. By using a continuity argument, it can be shown that the validity of this asymptotic formula can be extended to complex $y \in \mathbb{C}\backslash[-1, 1]$. Recall that $x_n = \sqrt{2n}y$. By the trapezoidal rule

$$\frac{1}{n}\sum_{k=1}^{n} f(k/n) = \int_0^1 f(t)dt + \frac{f(1) - f(0)}{2n} + O(n^{-2}),$$

we have

$$\frac{1}{n}\sum_{k=1}^{n} \log\frac{x_n + \sqrt{x_n^2-2k}}{x_n + \sqrt{x_n^2-2n}} = \frac{1}{n}\sum_{k=1}^{n} \log(y + \sqrt{y^2 - k/n}) - \log(y + \sqrt{y^2-1})$$

$$\sim \int_0^1 \log(y+\sqrt{y^2-t})dt + \frac{1}{2n}\log\frac{y+\sqrt{y^2-1}}{2y} - \log(y+\sqrt{y^2-1})$$

$$= y^2 - 1/2 - y\sqrt{y^2-1} + \frac{1}{2n}\log\frac{y+\sqrt{y^2-1}}{2y} \tag{3.6}$$

and

$$\sum_{k=1}^{n}\log\left[1+\frac{1}{2(x_n^2-2k)}\right] \sim \sum_{k=1}^{n}\frac{1}{2(x_n^2-2k)} = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{4(y^2-k/n)}$$

$$\sim \int_0^1 \frac{dt}{4(y^2-t)} = \frac{1}{4}\log\frac{y^2}{y^2-1} \tag{3.7}$$

as $n \to \infty$. Applying the last two formulas and (3.5) to (3.4) yields

$$\pi_n(x_n) \sim \prod_{k=1}^{n}\left[\frac{x_n+\sqrt{x_n^2-2n}}{2}\right]\cdot\prod_{k=1}^{n}\left[\frac{x_n+\sqrt{x_n^2-2k}}{x_n+\sqrt{x_n^2-2n}}\right]\cdot\prod_{k=1}^{n}\left[1+\frac{1}{2(x_n^2-2k)}\right]$$

$$\sim \left(\frac{n}{2}\right)^{n/2}(y+\sqrt{y^2-1})^n \exp[n(y^2-1/2-y\sqrt{y^2-1})]$$

$$\times \left(\frac{y+\sqrt{y^2-1}}{2y}\right)^{1/2}\left(\frac{y^2}{y^2-1}\right)^{1/4}$$

$$\sim \left(\frac{n}{2e}\right)^{n/2}\exp\{n[y^2-y\sqrt{y^2-1}+\log(y+\sqrt{y^2-1})]\}$$

$$\times \left(\frac{y+\sqrt{y^2-1}}{2\sqrt{y^2-1}}\right)^{1/2},$$

thus proving (3.3). □

Theorem 3.2. *Let $\delta > 0$ be any fixed small number. For y in a small complex neighborhood of $[-1+\delta, 1-\delta]$, we have*

$$\pi_n(\sqrt{2n}y) \sim \left(\frac{n}{2e}\right)^{n/2}\frac{e^{ny^2}}{(1-y^2)^{1/4}}$$

$$\times\left\{\cos\left[n(\theta-\sin\theta\cos\theta)+\frac{\theta}{2}\right]+\sin\left[n(\theta-\sin\theta\cos\theta)+\frac{\theta}{2}\right]\right\} \tag{3.8}$$

as $n \to \infty$, where $\theta = \theta(y) := \arccos y$.

To prove the above theorem, we need a lemma analogous to [8, Lemma 1]. For convenience, we use the notation

$$y_\pm := \left(\frac{n}{n\pm 1}\right)^{1/2} y \sim y \mp \frac{y}{2n} + \frac{3y}{8n^2}. \tag{3.9}$$

Lemma 3.3. *Let $\varphi(y)$ be any analytic function in a small complex neighborhood of $[-1+\delta, 1-\delta]$. We have*

$$\cos[(n\pm 1)\varphi(y_\pm)] \sim \cos(n\varphi)\left(\cos\lambda \mp \frac{\mu}{n}\sin\lambda\right) \mp \sin(n\varphi)\left(\sin\lambda \pm \frac{\mu}{n}\cos\lambda\right) \tag{3.10}$$

and

$$\sin[(n\pm 1)\varphi(y_\pm)] \sim \sin(n\varphi)\left(\cos\lambda \mp \frac{\mu}{n}\sin\lambda\right) \pm \cos(n\varphi)\left(\sin\lambda \pm \frac{\mu}{n}\cos\lambda\right) \tag{3.11}$$

as $n \to \infty$, where

$$\lambda = \lambda(y) := \varphi(y) - \frac{y\varphi'(y)}{2}, \tag{3.12}$$

and

$$\mu = \mu(y) := -\frac{y\varphi'(y)}{8} + \frac{y^2\varphi''(y)}{8}. \tag{3.13}$$

Proof. From (3.9) we have

$$\begin{aligned}(n\pm 1)\varphi(y_\pm) &\sim n\left(1 \pm \frac{1}{n}\right)\varphi\left(y \mp \frac{y}{2n} + \frac{3y}{8n^2}\right)\\ &\sim n\left(1 \pm \frac{1}{n}\right)\left(\varphi \mp \frac{y\varphi'}{2n} + \frac{3y\varphi'}{8n^2} + \frac{y^2\varphi''}{8n^2}\right)\\ &\sim n\left(\varphi \pm \frac{\lambda}{n} + \frac{\mu}{n^2}\right),\end{aligned}$$

where φ denotes $\varphi(y)$, and λ and μ are given in (3.12) and (3.13). It then follows that

$$\begin{aligned}\cos[(n\pm 1)\varphi(y_\pm)] &\sim \cos(n\varphi)\cos(\lambda \pm \mu/n) \mp \sin(n\varphi)\sin(\lambda \pm \mu/n)\\ &\sim \cos(n\varphi)\left(\cos\lambda \mp \frac{\mu}{n}\sin\lambda\right) \mp \sin(n\varphi)\left(\sin\lambda \pm \frac{\mu}{n}\cos\lambda\right);\\ \sin[(n\pm 1)\varphi(y_\pm)] &\sim \sin(n\varphi)\cos(\lambda \pm \mu/n) \pm \cos(n\varphi)\sin(\lambda \pm \mu/n)\\ &\sim \sin(n\varphi)\left(\cos\lambda \mp \frac{\mu}{n}\sin\lambda\right) \pm \cos(n\varphi)\left(\sin\lambda \pm \frac{\mu}{n}\cos\lambda\right).\end{aligned}$$

This proves the lemma. □

Proof of Theorem 3.2. Define

$$p_n(x) := [\Gamma(n/2 + 1/2)]^{-1}\pi_n(x). \tag{3.14}$$

We make a change of variable $x = x_n := \sqrt{2n}y$. It is easily seen from (3.1) and (3.14) that

$$\frac{\Gamma(n/2+1/2)\sqrt{2n}}{\Gamma(n/2+1)} \cdot y p_n(\sqrt{2n}y) = p_{n+1}(\sqrt{2n}y) + p_{n-1}(\sqrt{2n}y). \tag{3.15}$$

As in (2.19), we now assume

$$p_n(\sqrt{2n}y) \sim n^\alpha [r(y)]^n \{f(y)\cos[n\varphi(y)] + g(y)\sin[n\varphi(y)]\} \tag{3.16}$$

as $n \to \infty$. First, we shall determine the constant α and the functions $r(y)$ and $\varphi(y)$ in (3.16). From (3.9) and (3.16), we have

$$\begin{aligned} p_{n\pm1}(\sqrt{2n}y) = p_{n\pm1}(\sqrt{2(n\pm1)}y_\pm) &\sim (n\pm1)^\alpha [r(y_\pm)]^{n\pm1} \\ &\times \{f(y_\pm)\cos[(n\pm1)\varphi(y_\pm)] + g(y_\pm)\sin[(n\pm1)\varphi(y_\pm)]\}. \end{aligned} \tag{3.17}$$

Moreover, it can be shown from (3.9) that

$$[r(y_\pm)]^{n\pm1} \sim r^{n\pm1} e^{\mp yr'/2r}, \tag{3.18}$$

where $r = r(y)$. Applying (3.9)–(3.11) and (3.18) to (3.17) yields

$$\begin{aligned} p_{n\pm1}(\sqrt{2n}y) &\sim n^\alpha r^{n\pm1} e^{\mp yr'/2r} \\ &\times [(f\cos\lambda \pm g\sin\lambda)\cos(n\varphi) + (g\cos\lambda \mp f\sin\lambda)\sin(n\varphi)]. \end{aligned} \tag{3.19}$$

Here f and g stand for $f(y)$ and $g(y)$. By Stirling's formula [1, (6.1.37)] we have

$$\frac{\Gamma(n/2+1/2)\sqrt{n/2}}{\Gamma(n/2+1)} \sim 1 - \frac{1}{4n}. \tag{3.20}$$

A combination of (3.15), (3.16), (3.19) and (3.20) implies

$$\begin{aligned} &2y[f\cos(n\varphi) + g\sin(n\varphi)] \\ &\quad\sim re^{-yr'/2r}[(f\cos\lambda + g\sin\lambda)\cos(n\varphi) + (g\cos\lambda - f\sin\lambda)\sin(n\varphi)] \\ &\quad\quad + r^{-1}e^{yr'/2r}[(f\cos\lambda - g\sin\lambda)\cos(n\varphi) + (g\cos\lambda + f\sin\lambda)\sin(n\varphi)]. \end{aligned}$$

Comparing the coefficients of $\cos(n\varphi)$ and $\sin(n\varphi)$ on both sides of the last formula gives

$$\begin{aligned} 2yf &= re^{-yr'/2r}(f\cos\lambda + g\sin\lambda) + r^{-1}e^{yr'/2r}(f\cos\lambda - g\sin\lambda); \\ 2yg &= re^{-yr'/2r}(g\cos\lambda - f\sin\lambda) + r^{-1}e^{yr'/2r}(g\cos\lambda + f\sin\lambda). \end{aligned}$$

Solving the above two equations, we obtain

$$\cosh\left(\log r - \frac{yr'}{2r}\right)\cos\lambda = y, \quad \sinh\left(\log r - \frac{yr'}{2r}\right)\sin\lambda = 0.$$

A solution is

$$\log r - \frac{yr'}{2r} = 0, \quad \cos\lambda = y. \tag{3.21}$$

The first equation in (3.21) implies

$$r = e^{cy^2} \tag{3.22}$$

for some constant $c \in \mathbb{C}$. From (3.12) and (3.21), we have

$$\varphi = \pm(\arccos y - y\sqrt{1-y^2}) + c'y^2 + 2k\pi$$

for some constant $c' \in \mathbb{C}$ and $k \in \mathbb{N}$. Without loss of generality, we may take $c' = 0$ and $k = 0$. Hence,

$$\varphi = \arccos y - y\sqrt{1-y^2}. \tag{3.23}$$

Next, we are going to determine the functions f and g in (3.16). From (3.9) and (3.22), we obtain

$$[r(y_\pm)]^{n\pm 1} = [r(y)]^n \tag{3.24}$$

and

$$f(y_\pm) \sim f \mp \frac{yf'}{2n}, \quad g(y_\pm) \sim g \mp \frac{yg'}{2n}, \tag{3.25}$$

where $f = f(y)$ and $g = g(y)$. Applying (3.10), (3.11), (3.24) and (3.25) to (3.17) yields

$$\begin{aligned}\frac{p_{n\pm1}(\sqrt{2n}y)}{n^\alpha r^n} \sim {} & (f\cos\lambda \pm g\sin\lambda)\cos(n\varphi) + (g\cos\lambda \mp f\sin\lambda)\sin(n\varphi) \\ & + \frac{\cos(n\varphi)}{n}\left(\pm\alpha f\cos\lambda \mp \mu f\sin\lambda \mp \frac{yf'}{2}\cos\lambda\right. \\ & \left. + \alpha g\sin\lambda + \mu g\cos\lambda - \frac{yg'}{2}\sin\lambda\right) \\ & + \frac{\sin(n\varphi)}{n}\left(\pm\alpha g\cos\lambda \mp \mu g\sin\lambda \mp \frac{yg'}{2}\cos\lambda\right. \\ & \left. - \alpha f\sin\lambda - \mu f\cos\lambda + \frac{yf'}{2}\sin\lambda\right).\end{aligned} \tag{3.26}$$

A combination of (3.15), (3.16), (3.20) and (3.26) gives

$$\begin{aligned}&\left(1 - \frac{1}{4n}\right)[2yf\cos(n\varphi) + 2yg\,\sin(n\varphi)] \\ &\quad \sim 2f\cos\lambda\cos(n\varphi) + 2g\cos\lambda\sin(n\varphi)\end{aligned}$$

$$+\frac{\cos(n\varphi)}{n}(2\alpha g\sin\lambda+2\mu g\cos\lambda-yg'\sin\lambda)$$
$$+\frac{\sin(n\varphi)}{n}(-2\alpha f\sin\lambda-2\mu f\cos\lambda+yf'\sin\lambda).$$

In view of the second equation in (3.21), we obtain by matching the coefficients in the last formula

$$\frac{f}{2}\cos\lambda+2\alpha g\sin\lambda+2\mu g\cos\lambda-yg'\sin\lambda=0; \tag{3.27}$$
$$\frac{g}{2}\cos\lambda-2\alpha f\sin\lambda-2\mu f\cos\lambda+yf'\sin\lambda=0. \tag{3.28}$$

Note from (3.12), (3.13) and (3.23) that

$$\lambda=\arccos y,\quad \mu=\frac{y}{4\sqrt{1-y^2}}.$$

Hence, Eqs. (3.27) and (3.28) can be written as

$$f+\frac{4\alpha g\sqrt{1-y^2}}{y}+\frac{yg}{\sqrt{1-y^2}}-2g'\sqrt{1-y^2}=0; \tag{3.29}$$
$$g-\frac{4\alpha f\sqrt{1-y^2}}{y}-\frac{yf}{\sqrt{1-y^2}}+2f'\sqrt{1-y^2}=0. \tag{3.30}$$

Set

$$u:=y^{-2\alpha}(1-y^2)^{1/4}f;\quad v:=y^{-2\alpha}(1-y^2)^{1/4}g. \tag{3.31}$$

We then have from (3.29)–(3.31)

$$u'=-\frac{v}{2\sqrt{1-y^2}};\quad v'=\frac{u}{2\sqrt{1-y^2}}. \tag{3.32}$$

Define

$$\theta=\theta(y):=\arccos y. \tag{3.33}$$

The solution of the system (3.32) is given by

$$u=C_1\cos\frac{\theta}{2}+C_2\sin\frac{\theta}{2};\quad v=-C_1\sin\frac{\theta}{2}+C_2\cos\frac{\theta}{2}, \tag{3.34}$$

where $C_1\in\mathbb{C}$ and $C_2\in\mathbb{C}$ are two arbitrary constants. Consequently, we obtain from (3.31) that

$$f=\frac{y^{2\alpha}}{(1-y^2)^{1/4}}\left(C_1\cos\frac{\theta}{2}+C_2\sin\frac{\theta}{2}\right); \tag{3.35}$$
$$g=\frac{y^{2\alpha}}{(1-y^2)^{1/4}}\left(-C_1\sin\frac{\theta}{2}+C_2\cos\frac{\theta}{2}\right). \tag{3.36}$$

Applying (3.22), (3.35) and (3.36) to (3.16) yields

$$p_n(\sqrt{2n}y) \sim n^\alpha e^{ncy^2} y^{2\alpha}(1-y^2)^{-1/4}[C_1\cos(n\varphi+\theta/2)+C_2\sin(n\varphi+\theta/2)]. \tag{3.37}$$

This formula holds uniformly for y in a small complex neighborhood of $[-1+\delta, 1-\delta]$. Moreover, it follows from (3.3) and (3.14) that

$$p_n(\sqrt{2n}y) \sim \frac{1}{\sqrt{2\pi}}\exp\{n[y^2 - y\sqrt{y^2-1} + \log(y+\sqrt{y^2-1})]\}\left(\frac{y+\sqrt{y^2-1}}{2\sqrt{y^2-1}}\right)^{1/2} \tag{3.38}$$

for complex y bounded away from $[-1,1]$. Finally, we match the above two formulas in an overlapping region to determine the constants α, c, C_1 and C_2 in (3.37). For $\operatorname{Im} y > 0$, it follows from (3.33) that $\operatorname{Im}\theta < 0$; see a similar statement following (2.28). Furthermore, it can be shown from (3.23) that if $\operatorname{Im} y > 0$, then we also have $\operatorname{Im}\varphi < 0$. (To do this, one first notes that $\varphi'(y)$ is negative for $y \in [-1+\delta, 1-\delta]$. Then, by the continuity of φ', one concludes that $\operatorname{Re}\varphi'(y) < 0$ for y in a neighborhood of $[-1+\delta, 1-\delta]$ in the complex plane. Finally, the mean value theorem ensures that there exists a real number $\xi \in (0, \operatorname{Im} y)$ such that $\varphi(y) = \varphi(\operatorname{Re} y) + i(\operatorname{Im} y)\varphi'(\operatorname{Re} y + i\xi)$, from which one obtains $\operatorname{Im}\varphi(y) < 0$.) Thus, (3.37) implies

$$p_n(\sqrt{2n}y) \sim n^\alpha e^{ncy^2} y^{2\alpha}(1-y^2)^{-1/4}\left(\frac{C_1}{2}+\frac{C_2}{2i}\right)e^{in\varphi+i\theta/2}.$$

Meanwhile, we have from (3.33) and (3.38)

$$p_n(\sqrt{2n}y) \sim \frac{1}{\sqrt{2\pi}}\exp\{n[y^2 - iy\sqrt{1-y^2} + i\arccos y]\}\left[\frac{e^{i(\theta-\pi/2)}}{2\sqrt{1-y^2}}\right]^{1/2}.$$

Thus, we obtain from (3.23) and the above two formulas that $\alpha = 0$, $c = 1$ and

$$\frac{C_1}{2}+\frac{C_2}{2i} = \frac{e^{-i\pi/4}}{2\sqrt{\pi}}.$$

Similarly, matching (3.37) with (3.38) in the region $\operatorname{Im} y < 0$ yields again $\alpha = 0$, $c = 1$ and the equation

$$\frac{C_1}{2}-\frac{C_2}{2i} = \frac{e^{i\pi/4}}{2\sqrt{\pi}}.$$

Coupling the last two equations gives

$$C_1 = C_2 = \frac{1}{\sqrt{2\pi}}.$$

Therefore, we conclude that

$$\alpha = 0, \quad c = 1, \quad C_1 = C_2 = \frac{1}{\sqrt{2\pi}}.$$

This, together with (3.14), (3.23) and (3.37), yields (3.8). □

4. Case 3: An Open Problem

Recently, Ismail proposed the problem of finding asymptotic formulas for the orthogonal polynomials determined by

$$\pi_{n+1}(x) = (x - n^2)\pi_n(x) - \frac{1}{4}\pi_{n-1}(x), \quad n \geq 1, \tag{4.1}$$

$$\pi_0(x) = 1, \quad \pi_1(x) = x; \tag{4.2}$$

see [5, Sec. 6] and [4, p. 370]. We first present a result for x not in the interval of oscillation.

Theorem 4.1. *As $n \to \infty$, we have*

$$\pi_n(n^2 y) \sim \left(\frac{n}{e}\right)^{2n} \exp\{n[(\sqrt{y}+1)\log(\sqrt{y}+1) - (\sqrt{y}-1)\log(\sqrt{y}-1)]\} \left(\frac{y}{y-1}\right)^{1/2} \tag{4.3}$$

for complex y bounded away from $[0,1]$.

Proof. Set

$$\pi_n(x) = \prod_{k=1}^{n} w_k(x). \tag{4.4}$$

It follows from (4.1) and (4.2) that $w_1(x) = x$ and

$$w_{k+1}(x) = x - k^2 - \frac{1}{4w_k(x)}.$$

Let $x = x_n := n^2 y$ with $y \in \mathbb{C}\backslash[0,1]$. As with the case of Hermite polynomials, it can be shown that for real x and $x \notin [0, n^2]$, we have

$$x - (k-1)^2 - 1 < w_k(x) < x - (k-1)^2 + 1$$

for all $k = 1, \ldots, n$. Thus,

$$1 + \frac{2k}{x-k^2} - \frac{2}{x-k^2} < \frac{w_k(x)}{x-k^2} < 1 + \frac{2k}{x-k^2}.$$

Consequently,

$$w_k(n^2 y) = n^2\left(y - \frac{k^2}{n^2}\right)\left[1 + \frac{2k}{n^2 y - k^2} + O(n^{-2})\right] \tag{4.5}$$

as $n \to \infty$, uniformly in $k = 1, \ldots, n$. By using a continuity argument, it can be shown that the validity of this asymptotic formula can be extended to complex

$y \in \mathbb{C}\backslash[0,1]$. In view of the trapezoidal rule

$$\frac{1}{n}\sum_{k=1}^{n} f(k/n) \sim \int_0^1 f(t)dt + \frac{f(1)-f(0)}{2n},$$

we have

$$\begin{aligned}\sum_{k=1}^{n} \log\left(y - \frac{k^2}{n^2}\right) &\sim n\int_0^1 \log(y-t^2)dt + \frac{1}{2}\log\frac{y-1}{y}\\ &= n[(\sqrt{y}+1)\log(\sqrt{y}+1) - (\sqrt{y}-1)\log(\sqrt{y}-1) - 2]\\ &\quad + \frac{1}{2}\log\frac{y-1}{y}\end{aligned}$$

and

$$\sum_{k=1}^{n} \log\left(1 + \frac{2k}{n^2y-k^2}\right) \sim \sum_{k=1}^{n} \frac{2k}{n^2y-k^2} \sim \int_0^1 \frac{2t}{y-t^2}dt = \log\frac{y}{y-1}$$

as $n \to \infty$. Applying the last two formulas and (4.5) to (4.4) gives (4.3). □

Next, we give a result for x inside the interval of oscillation.

Theorem 4.2. *Let $\delta > 0$ be any fixed small number. For y in a small neighborhood of $[\delta, 1-\delta]$ in the complex plane, we have*

$$\pi_n(n^2y) \sim (-1)^{n-1} 2\sin(n\pi\sqrt{y})\left(\frac{n}{e}\right)^{2n}\left(\frac{1+\sqrt{y}}{1-\sqrt{y}}\right)^{n\sqrt{y}} y^{1/2}(1-y)^{n-1/2} \quad (4.6)$$

as $n \to \infty$.

To prove the above theorem, we will need a lemma analogous to [8, Lemma 1]. As in (3.9), for convenience we set

$$y_\pm := \left(\frac{n}{n\pm 1}\right)^2 y \sim y \mp \frac{2y}{n} + \frac{3y}{n^2}. \quad (4.7)$$

Lemma 4.3. *Let $\varphi(y)$ be any analytic function in a small neighborhood of $[\delta, 1-\delta]$ in the complex plane. We have*

$$\cos[(n\pm 1)\varphi(y_\pm)] \sim \cos(n\varphi)\left(\cos\lambda \mp \frac{\mu}{n}\sin\lambda\right) \mp \sin(n\varphi)\left(\sin\lambda \pm \frac{\mu}{n}\cos\lambda\right) \quad (4.8)$$

and

$$\sin[(n\pm 1)\varphi(y_\pm)] \sim \sin(n\varphi)\left(\cos\lambda \mp \frac{\mu}{n}\sin\lambda\right) \pm \cos(n\varphi)\left(\sin\lambda \pm \frac{\mu}{n}\cos\lambda\right) \quad (4.9)$$

as $n \to \infty$, *where*

$$\lambda = \lambda(y) := \varphi(y) - 2y\varphi'(y), \tag{4.10}$$

and

$$\mu = \mu(y) := y\varphi'(y) + 2y^2\varphi''(y). \tag{4.11}$$

Proof. From (4.7) we have

$$\begin{aligned}(n \pm 1)\varphi(y_\pm) &\sim n\left(1 \pm \frac{1}{n}\right)\varphi\left(y \mp \frac{2y}{n} + \frac{3y}{n^2}\right)\\ &\sim n\left(1 \pm \frac{1}{n}\right)\left(\varphi \mp \frac{2y\varphi'}{n} + \frac{3y\varphi'}{n^2} + \frac{2y^2\varphi''}{n^2}\right)\\ &\sim n\left(\varphi \pm \frac{\lambda}{n} + \frac{\mu}{n^2}\right),\end{aligned}$$

where λ and μ are given in (4.10) and (4.11). It then follows that

$$\begin{aligned}\cos[(n \pm 1)\varphi(y_\pm)] &\sim \cos(n\varphi)\cos(\lambda \pm \mu/n) \mp \sin(n\varphi)\sin(\lambda \pm \mu/n)\\ &\sim \cos(n\varphi)\left(\cos\lambda \mp \frac{\mu}{n}\sin\lambda\right) \mp \sin(n\varphi)\left(\sin\lambda \pm \frac{\mu}{n}\cos\lambda\right);\\ \sin[(n \pm 1)\varphi(y_\pm)] &\sim \sin(n\varphi)\cos(\lambda \pm \mu/n) \pm \cos(n\varphi)\sin(\lambda \pm \mu/n)\\ &\sim \sin(n\varphi)\left(\cos\lambda \mp \frac{\mu}{n}\sin\lambda\right) \pm \cos(n\varphi)\left(\sin\lambda \pm \frac{\mu}{n}\cos\lambda\right).\end{aligned}$$

This proves the lemma. □

Proof of Theorem 4.2. Define

$$p_n(x) := \frac{(-1)^n}{\Gamma(n)^2}\pi_n(x). \tag{4.12}$$

We make a change of variable $x = x_n := n^2 y$. It is readily seen from (4.1) and (4.12) that

$$(1 - y)p_n(n^2y) = p_{n+1}(n^2y) + \frac{1}{4n^2(n-1)^2}p_{n-1}(n^2y). \tag{4.13}$$

As in (3.16), we first assume

$$p_n(n^2y) \sim n^\alpha[r(y)]^n\{f(y)\cos[n\varphi(y)] + g(y)\sin[n\varphi(y)]\} \tag{4.14}$$

as $n \to \infty$, and then determine the constant α and the functions $r(y)$, $f(y)$, $g(y)$ and $\varphi(y)$ in the formula. From (4.7) and (4.14) we have

$$\begin{aligned}p_{n\pm1}(n^2y) &= p_{n\pm1}((n \pm 1)^2 y_\pm)\\ &\sim (n \pm 1)^\alpha[r(y_\pm)]^{n\pm1}\{f(y_\pm)\cos[(n \pm 1)\varphi(y_\pm)]\\ &\quad + g(y_\pm)\sin[(n \pm 1)\varphi(y_\pm)]\}.\end{aligned} \tag{4.15}$$

Moreover, it can be shown from (4.7) that as $n \to \infty$, we also have

$$[r(y_\pm)]^{n\pm1} \sim r^{n\pm1} e^{\mp 2yr'/r}, \tag{4.16}$$

where r stands for $r(y)$. Applying (4.7)–(4.9) and (4.16) to (4.15) yields

$$p_{n\pm1}(n^2 y) \sim n^\alpha r^{n\pm1} e^{\mp 2yr'/r} \\ \times [(f\cos\lambda \pm g\sin\lambda)\cos(n\varphi) + (g\cos\lambda \mp f\sin\lambda)\sin(n\varphi)]. \tag{4.17}$$

A combination of (4.13), (4.14) and (4.17) gives

$$(1-y)[f\cos(n\varphi) + g\sin(n\varphi)] \sim re^{-2yr'/r} \\ \times [(f\cos\lambda + g\sin\lambda)\cos(n\varphi) + (g\cos\lambda - f\sin\lambda)\sin(n\varphi)].$$

By comparing the coefficients of $\cos(n\varphi)$ and $\sin(n\varphi)$ on both sides of the last formula, we obtain

$$(1-y)f = re^{-2yr'/r}(f\cos\lambda + g\sin\lambda);$$
$$(1-y)g = re^{-2yr'/r}(g\cos\lambda - f\sin\lambda).$$

Thus, we have from the above equations

$$(1-y) = re^{-2yr'/r}\cos\lambda, \quad 0 = re^{-2yr'/r}\sin\lambda.$$

The only solution is $\lambda = 0$, and

$$re^{-2yr'/r} = 1 - y. \tag{4.18}$$

With $\lambda = 0$, we obtain from (4.10)

$$\varphi = c\sqrt{y} \tag{4.19}$$

for some constant $c \in \mathbb{C}$. Let $R(y) := \log r(y)$. From (4.18), it is easily seen that $R(y)$ satisfies a first-order linear inhomogeneous equation, whose solution is given by

$$R(y) = -\frac{1}{2}y^{1/2}\left[\int^y s^{-3/2}\log(1-s)ds\right].$$

Upon integration by parts, followed by a change of variable $u = s^{1/2}$, one obtains

$$R(y) = \log(1-y) + 2y^{1/2}\text{arctanh}\sqrt{y} + c'\sqrt{y}$$

for some constant $c' \in \mathbb{C}$. Taking exponentials on both sides gives

$$r(y) = (1-y)\left(\frac{1+\sqrt{y}}{1-\sqrt{y}}\right)^{\sqrt{y}} e^{c'\sqrt{y}}.$$

Without loss of generality, we may take $c' = 0$. Hence,

$$r(y) = (1-y)\left(\frac{1+\sqrt{y}}{1-\sqrt{y}}\right)^{\sqrt{y}}. \tag{4.20}$$

Next, we determine the functions f and g in (4.14). From (4.7), (4.18) and (4.20) we have

$$[r(y_\pm)]^{n\pm 1} \sim (1-y)^{\pm 1}[r(y)]^n \left[1 + \frac{y}{n(1-y)}\right]. \tag{4.21}$$

Furthermore, it is easily seen from (4.7) and (4.19) that $(n\pm 1)\varphi(y_\pm) \sim n\varphi(y)$ and

$$f(y_\pm) \sim f(y) \mp \frac{2yf'(y)}{n}, \quad g(y_\pm) \sim g(y) \mp \frac{2yg'(y)}{n}.$$

Applying the above formulas for functions r, φ, f and g to (4.15) yields

$$\begin{aligned} p_{n\pm 1}(n^2y) &\sim n^\alpha r^n (1-y)^{\pm 1} \\ &\times \left(1 \pm \frac{\alpha}{n}\right)\left[1 + \frac{y}{n(1-y)}\right]\left[\left(f \mp \frac{2yf'}{n}\right)\cos(n\varphi) + \left(g \mp \frac{2yg'}{n}\right)\sin(n\varphi)\right]. \end{aligned}$$

(One can also obtain this result from Lemma 4.3, since $\lambda = \mu = 0$ by (4.19).) This, together with (4.13) and (4.14), implies

$$\begin{aligned} f\cos(n\varphi) + g\sin(n\varphi) &\sim \left[f + \frac{1}{n}\left(\frac{yf}{1-y} + \alpha f - 2yf'\right)\right] \\ &\times \cos(n\varphi) + \left[g + \frac{1}{n}\left(\frac{yg}{1-y} + \alpha g - 2yg'\right)\right]\sin(n\varphi). \end{aligned}$$

Comparing the coefficients on both sides of the last formula gives

$$\frac{yf}{1-y} + \alpha f - 2yf' = 0, \quad \frac{yg}{1-y} + \alpha g - 2yg' = 0.$$

Hence,

$$f = C_1 y^{\alpha/2}(1-y)^{-1/2}, \quad g = C_2 y^{\alpha/2}(1-y)^{-1/2}, \tag{4.22}$$

where $C_1 \in \mathbb{C}$ and $C_2 \in \mathbb{C}$ are two arbitrary constants. Applying (4.19), (4.20) and (4.22) to (4.14) yields

$$p_n(n^2y) \sim n^\alpha y^{\alpha/2}(1-y)^{n-1/2}\left(\frac{1+\sqrt{y}}{1-\sqrt{y}}\right)^{n\sqrt{y}} [C_1\cos(nc\sqrt{y}) + C_2\sin(nc\sqrt{y})]. \tag{4.23}$$

This formula holds uniformly for y in a small neighborhood of $[\delta, 1-\delta]$ in the complex plane. Moreover, it follows from (4.3) and (4.12) that

$$\begin{aligned} p_n(n^2y) \sim \frac{(-1)^n n}{2\pi} &\exp\{n[(\sqrt{y}+1)\log(\sqrt{y}+1) \\ &- (\sqrt{y}-1)\log(\sqrt{y}-1)]\}\left(\frac{y}{y-1}\right)^{1/2} \end{aligned} \tag{4.24}$$

for complex y bounded away from $[0, 1]$. At the final stage, we match the last two formulas in an overlapping region to determine the constants α, c, C_1 and C_2 in

(4.23). In view of the equalities $\exp(\pm inc\sqrt{y}) = \cos(nc\sqrt{y}) \pm i\sin(nc\sqrt{y})$ and

$$(1-y)^n \left(\frac{1+\sqrt{y}}{1-\sqrt{y}}\right)^{n\sqrt{y}} = \exp\{n[(\sqrt{y}+1)\log(\sqrt{y}+1) - (\sqrt{y}-1)\log(1-\sqrt{y})]\},$$

formula (4.23) can be written as

$$p_n(n^2y) \sim n^\alpha y^{\alpha/2}(1-y)^{-1/2}\exp\{n[(\sqrt{y}+1)\log(\sqrt{y}+1) - (\sqrt{y}-1)\log(1-\sqrt{y})]\} \times \left[\left(\frac{C_1}{2} - \frac{C_2}{2i}\right)e^{-inc\sqrt{y}} + \left(\frac{C_1}{2} + \frac{C_2}{2i}\right)e^{inc\sqrt{y}}\right]. \tag{4.25}$$

Meanwhile, it follows from (4.24) that for $\operatorname{Im} y > 0$, we have

$$p_n(n^2y) \sim \frac{n}{2\pi}\exp\{n[(\sqrt{y}+1)\log(\sqrt{y}+1) - (\sqrt{y}-1)\log(1-\sqrt{y})] - in\pi\sqrt{y} - i\pi/2\}\left(\frac{y}{1-y}\right)^{1/2}.$$

A comparison of the above two asymptotic formulas shows that $\alpha = 1$ and $c = \pi$ or $c = -\pi$. Without loss of generality, we take $c = \pi$. Note that the function $\exp(inc\sqrt{y}) = \exp(in\pi\sqrt{y})$ is exponentially small, and hence negligible in the region $\operatorname{Im} y > 0$. By matching the last two formulas one more time, and ignoring the exponentially small term, we have

$$\frac{C_1}{2} - \frac{C_2}{2i} = \frac{e^{-i\pi/2}}{2\pi}.$$

With $\alpha = 1$ and $c = \pi$, we match (4.24) with (4.25) in the region $\operatorname{Im} y < 0$ to obtain the other equation

$$\frac{C_1}{2} + \frac{C_2}{2i} = \frac{e^{i\pi/2}}{2\pi}.$$

Upon solving the last two equations, we obtain $C_1 = 0$ and $C_2 = -1/\pi$. Therefore, we conclude that

$$\alpha = 1, \quad c = \pi, \quad C_1 = 0, \quad C_2 = -1/\pi.$$

Combining this with (4.12) and (4.23) gives (4.6). □

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables* (Dover Publications, Inc., New York, 1970).

[2] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* **52** (1999) 1335–1425.

[3] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* **52** (1999) 1491–1552.

[4] S. S. Goh and C. A. Micchelli, Uncertainty principles in Hilbert spaces, *J. Fourier Anal. Appl.* **8** (2002) 335–373.

[5] M. E. H. Ismail and E. Koelink, The J-Matrix method and eigenfunction expansions, preprint (July, 2010).

[6] R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report no. 98-17, TU-Delft (1998).

[7] F. W. J. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974); Reprinted by A. K. Peters, Wellesley, MA (1997).

[8] Z. Wang and R. Wong, Uniform asymptotic expansion of $J_\nu(\nu a)$ via a difference equation, *Numer. Math.* **91** (2002) 147–193.

[9] R. Wong, *Asymptotic Approximations of Integrals* (Academic Press, Boston, 1989); Reprinted by SIAM, Philadelphia, PA (2001).

[10] R. Wong and H. Li, Asymptotic expansions for second-order linear difference equations, *J. Comput. Appl. Math.* **41** (1992) 65–94.