Analysis and Applications, Vol. 18, No. 2 (2020) 237–270
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DOI: 10.1142/S0219530519500076

Asymptotics of the Wilson polynomials

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Received 18 January 2019
Accepted 29 April 2019
Published 13 June 2019

In this paper, we study the asymptotic behavior of the Wilson polynomials $W_n(x; a, b, c, d)$ as their degree tends to infinity. These polynomials lie on the top level of the Askey scheme of hypergeometric orthogonal polynomials. Infinite asymptotic expansions are derived for these polynomials in various cases, for instance, (i) when the variable x is fixed and (ii) when the variable is rescaled as $x = n^2 t$ with $t \geq 0$. Case (ii) has two subcases, namely, (a) zero-free zone ($t > 1$) and (b) oscillatory region ($0 < t < 1$). Corresponding results are also obtained in these cases (iii) when t lies in a neighborhood of the transition point $t = 1$, and (iv) when t is in the neighborhood of the transition point $t = 0$. The expansions in the last two cases hold uniformly in t. Case (iv) is also the only unsettled case in a sequence of works on the asymptotic analysis of linear difference equations.

Keywords: Three-term recurrence relations; transition point; Wilson polynomials; uniform asymptotics; zeros.

Mathematics Subject Classification 2020: 41A60, 33C45, 39A06

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1. Introduction

In 1980, Wilson [29] introduced a sequence of orthogonal polynomials defined by the hypergeometric function

$$\frac{W_n(x;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n} = {}_4F_3\left(\begin{matrix}-n, n+a+b+c+d-1, a+iy, a-iy \\ a+b, a+c, a+d\end{matrix}; 1\right), \tag{1.1}$$

where $x = y^2$ and $(a)_n = \Gamma(n+a)/\Gamma(a)$ is the Pochhammer symbol. These polynomials are now known as the Wilson polynomials, and they lie on the top level of the Askey scheme of hypergeometric orthogonal polynomials; see [16]. Since

$$(a+iy)_k(a-iy)_k = \prod_{j=0}^{k-1}[(a+j)^2 + x],$$

it is easily seen that $W_n(x;a,b,c,d)$ is a polynomial of degree n in the variable x. If a,b,c,d are all positive real numbers or $c = \bar{a}$, $d = \bar{b}$ and $\mathrm{Re}\,(a,b,c,d) > 0$, these polynomials are orthogonal on the half-line $(0,\infty)$ with respect to the weight function [21, (18.25.4)]

$$w(x) = \frac{1}{2y}\left|\frac{\Gamma(a+iy)\Gamma(b+iy)\Gamma(c+iy)\Gamma(d+iy)}{\Gamma(2iy)}\right|^2, \quad x = y^2. \tag{1.2}$$

From (1.1), one can show that the leading coefficient of $W_n(x;a,b,c,d)$ is

$$\gamma_n = (-1)^n(n+a+b+c+d-1)_n. \tag{1.3}$$

Consider the monic Wilson polynomials $\{\pi_n(x)\}$ given by

$$\pi_n(x) = \frac{W_n(x;a,b,c,d)}{\gamma_n} = \frac{W_n(x;a,b,c,d)}{(-1)^n(n+a+b+c+d-1)_n}; \tag{1.4}$$

they satisfy the recurrence relation

$$x\pi_n(x) = \pi_{n+1}(x) + (A_n + C_n - a^2)\pi_n(x) + A_{n-1}C_n\pi_{n-1}(x), \tag{1.5}$$

where

$$A_n = \frac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)} \tag{1.6}$$

and

$$C_n = \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}; \tag{1.7}$$

see [16, (9.1.5), p. 187]. The orthogonality relations of Wilson polynomials can be obtained via local symmetry (recurrence relation) techniques [20]. The Wilson polynomials are also related to the generic superintegrable system on the 2-sphere [15].

In this paper, we are interested in the asymptotic behavior of the Wilson polynomials as the degree n grows to infinity. For fixed x, Wilson [30] has already given the large n asymptotics of $W_n(x)$. However, numerical simulations indicate that the asymptotic formulas in [30] need some modifications. Moreover, we are also interested in other types of asymptotic approximations, such as uniform asymptotic formulas in the neighborhoods of the smallest and largest zeros, or formulas for x on the interval of orthogonality.

There are several existing techniques for deriving asymptotic approximations for a given sequence of orthogonal polynomials. For instance, the steepest descent method for integrals, the WKB approximations for differential equations, and the more recently introduced nonlinear steepest descent method for Riemann–Hilbert problems. For details of these approaches, we refer to [8, 21, 31]. However, the above-mentioned approaches all have their limitations. For example, the Wilson polynomials $W_n(x)$ with $x = y^2$ have the following generating function [22, (18.26.18)]:

$$ {}_2F_1\left(\begin{matrix} a+iy, d+iy \\ a+d \end{matrix}; t\right) {}_2F_1\left(\begin{matrix} b-iy, c-iy \\ b+c \end{matrix}; t\right) = \sum_{n=0}^{\infty} \frac{W_n(x; a, b, c, d)}{(a+d)_n (b+c)_n n!} t^n \tag{1.8} $$

for $|t| < 1$ from which one can derive a Cauchy integral representation for the Wilson polynomials. However, the resulting integral is too complicated to be analyzed. Also, since there is no known differential equation for the Wilson polynomials, none of the asymptotic tools available for differential equations can be applied. Furthermore, the recently introduced Riemann–Hilbert approach depends highly on the analyticity of the weight function. Since the weight function $w(x)$ in (1.2) involves Gamma functions which have poles, it makes the problem of deforming contour (or opening lenses) in the Riemann–Hilbert approach difficult. One may let the parameters a, b, c, d depend on n so that all the poles are pushed away from the real axis, like what Wang *et al.* [25] had done for the Meixner–Pollaczek polynomials. But, by doing this, one has changed the original problem into a varying weight problem, which is a different problem. Despite this possibility, there are four Gamma functions in the numerator of the weight function $w(x)$ in (1.2); comparing with the Meixner–Pollaczek polynomials whose weight function has only one Gamma function, one can imagine the complexity of the present situation.

On the other hand, we find that the three-term recurrence relation in (1.5) for the Wilson polynomials is not too complicated. In fact, each sequence of hypergeometric polynomials that are orthogonal on (an interval of) the real line satisfies a three-term recurrence relation, and these relations have relatively simple coefficients. Therefore, it seems that a natural approach to obtain asymptotic information for these polynomials is to analyze their three-term recurrence relations, which are just second-order linear difference equations. Work on asymptotic theory for difference equations dates back to Birkhoff [2, 3] and Adams [1]. Since then, several authors have made important contributions to this area, and applications of their results to orthogonal polynomials have turned out to be a great success. Among the works of these people, here we can mention a few, e.g., [5–7, 9–13, 17, 18, 24, 27, 28, 32]. In particular, we point out that in [11], Geronimo has already derived an Airy-type approximation for the Wilson polynomials near the large transition point.

The purpose of this paper is to use techniques developed for difference equations to obtain asymptotic expansions of the Wilson polynomials. These include fixed-x, scaled-x, and uniform asymptotics, as well as the asymptotics of their zeros as $n \to \infty$. We first recall the definition of transition points in [27, 28]. Consider a three-term recurrence relation

$$P_{n+1}(x) + P_{n-1}(x) = (\widetilde{A}_n x + \widetilde{B}_n) P_n(x) \tag{1.9}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1/K_0$, where K_0 is a constant and the recurrence coefficients have the following asymptotic expansions:

$$\widetilde{A}_n \sim \frac{1}{n^\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \quad \widetilde{B}_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s} \tag{1.10}$$

as $n \to \infty$. By the definition given in Wang and Wong [27], the *characteristic equation* associated to (1.9) is

$$\lambda^2 - (\alpha_0 t + \beta_0)\lambda + 1 = 0, \tag{1.11}$$

where we have scaled the variable $x = n^\theta t$. The *transition points* are the points where the roots of equation (1.11) coincide, i.e. when t satisfies

$$\alpha_0 t + \beta_0 = \pm 2. \tag{1.12}$$

According to the classification given in [4, 27, 28], there are three cases to be considered:

Case (i) $\theta \neq 0$ and the transition point is not equal to zero (or equivalently, not at the origin). This case was investigated in [27], and the results were Airy-type expansions valid in a region containing the transition point.

Case (ii) $\theta \neq 0$ and the transition point is equal to zero. This case was studied and partially solved in [4]. Under an additional assumption that $\theta \neq 2$, Bessel-type expansions were derived in a region containing the transition point at the origin.

Case (iii) $\theta = 0$. This case was considered in [28], and the uniform asymptotic formulas are given in terms of Bessel functions.

In the analysis of Case (ii) given in [4], the special value $\theta = 2$ was excluded, since $(\theta - 2)$ appears in the denominator of the order of the Bessel functions, cf. [4, Eq. (2.15)]. One of the objectives of the present work is to consider the exceptional case when $\theta = 2$ and the transition point is equal to zero, thus completing our investigation of the uniform asymptotic theory of linear difference equations.

Recall the monic Wilson polynomials (1.4) and their recurrence relation (1.5). We set

$$P_n(x) = \frac{\pi_n(x)}{K_n} \tag{1.13}$$

with

$$K_n := \frac{\Gamma_n(u-1, a+b, a+c, a+d, b+c, b+d, c+d, 1)}{\Gamma_n(u/2-1/2, u/2, u/2, u/2+1/2)}, \tag{1.14}$$

where, for simplicity, we denote $\Gamma_n(a) = \Gamma(n/2 + a/2)$, $\Gamma_n(a_1, \ldots, a_k) = \Gamma_n(a_1) \cdots \Gamma_n(a_k)$ and $u = a+b+c+d$. Note that $\Gamma_{n+1}(a)/\Gamma_{n-1}(a) = (n-1+a)/2$. It is readily verified that

$$\frac{K_{n+1}}{K_{n-1}} = A_{n-1}C_n, \tag{1.15}$$

and the new polynomials $P_n(x)$ satisfy the three-term recurrence relation (1.9) with the recurrence coefficients given by

$$\widetilde{A}_n = \frac{K_n}{K_{n+1}} = \frac{1}{n^2}\sum_{s=0}^{\infty}\frac{\alpha_s}{n^s}, \quad \widetilde{B}_n = \frac{K_n}{K_{n+1}}(a^2 - A_n - C_n) = \sum_{s=0}^{\infty}\frac{\beta_s}{n^s}. \tag{1.16}$$

The first few coefficients in the above expansions are

$$\alpha_0 = 4, \quad \alpha_1 = -4(a+b+c+d-1), \quad \beta_0 = -2, \quad \beta_1 = 0, \quad \beta_2 = \frac{1}{4}. \tag{1.17}$$

From (1.12), it is evident that there are two transition points $t_1 = 0$ and $t_2 = 1$. The transition point $t_2 = 1$ belongs to Case (i). The transition point $t_1 = 0$ belongs to the exceptional case not covered in [4]. Our purpose here is to fill the gap in the analysis of Case (ii), and to apply the result specifically to the Wilson polynomials.

This paper is organized as follows. In Sec. 2, we consider the fixed-x asymptotics by using the difference equation methods developed by Wong and Li [33, 34]. In Sec. 3, we apply Wang and Wong's method [26] to obtain the scaled-x asymptotic approximations of $W_n(x)$ in both oscillatory and zero-free regions. In Sec. 4, we provide a result on the general solution of (1.9) in the exceptional case. The uniform asymptotic formulas of $W_n(x)$ are then established in Sec. 5. Based on the uniform asymptotics, the Plancherel–Rotach-type asymptotics and asymptotics of the zeros are presented in Sec. 6.

2. Fixed-x Asymptotic Approximation

For convenience, we shall use the notation

$$u := a+b+c+d. \tag{2.1}$$

Note that the four parameters a, b, c and d are either real or complex-conjugate pairs, that is, u is a real number. By Stirling's formula, it follows from (1.3) and (1.14) that

$$\gamma_n = (-1)^n \Gamma(2n+u-1)/\Gamma(n+u-1) \sim (-1)^n 2^{u-3/2+2n}(n/e)^n \tag{2.2}$$

and

$$K_n \sim (n/2)^u (4\pi/n)^2 (n/(2e))^{2n} \sim 2^{4-u-2n}\pi^2 e^{-2n} n^{u-2+2n}. \tag{2.3}$$

For the Wilson polynomials, we have the following asymptotic approximations.

Theorem 1 (Fixed-x asymptotics). *Let*

$$\mathbf{C}_n = (2\pi)^{3/2} e^{-3n} n^{3n+u-3/2} \tag{2.4}$$

and

$$\mathbf{A}(z) = \frac{\Gamma(2z)}{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d+z)}. \tag{2.5}$$

For any fixed complex x such that $-2i\sqrt{x} \notin \mathbb{Z}$ and $\mathbf{A}(\pm i\sqrt{x}) \neq 0$, we have

$$W_n(x) \sim \mathbf{C}_n \left\{ \mathbf{A}(i\sqrt{x}) e^{i2\sqrt{x}\log n} \sum_{s=0}^{\infty} \frac{c_{s,1}}{n^s} + \mathbf{A}(-i\sqrt{x}) e^{-i2\sqrt{x}\log n} \sum_{s=0}^{\infty} \frac{c_{s,2}}{n^s} \right\} \tag{2.6}$$

as $n \to \infty$. When $-2i\sqrt{x} = k \in \mathbb{N}$ and $\mathbf{A}(-i\sqrt{x}) \neq 0$, we have

$$W_n(x) \sim \mathbf{C}_n \mathbf{A}(-i\sqrt{x}) \exp\{-i2\sqrt{x}\log n\} \left[\sum_{s=0}^{\infty} \frac{\bar{c}_{s,2}}{n^s} + \bar{c}(\log n) \sum_{s=0}^{\infty} \frac{c_{s,1}}{n^{s+2k}} \right] \tag{2.7}$$

as $n \to \infty$. Here, we have chosen $\sqrt{x} = i\sqrt{-x}$ if $x < 0$. In (2.6) and (2.7), $\bar{c}_{2k,2} = 0$, $c_{0,1} = c_{0,2} = \bar{c}_{0,2} = 1$ and the other coefficients can be determined recursively.

To derive an infinite asymptotic expansion for the Wilson polynomials when the variable is fixed, we shall make use of the results given in Wong and Li [33]. Let $y(n) = P_n(x)$, where $P_n(x)$ is the Wilson polynomials given in (1.13). The three-term recurrence relation in (1.9) can then be written as

$$y(n+2) + a(n)y(n+1) + b(n)y(n) = 0, \tag{2.8}$$

where

$$a(n) = -(\widetilde{A}_{n+1}x + \widetilde{B}_{n+1}) = 2 - \frac{4x + \frac{1}{4}}{n^2} + O(n^{-3}), \quad b(n) = 1; \tag{2.9}$$

cf. (1.17). In the notations of [33],

$$a(n) := \sum_{s=0}^{\infty} \frac{a_s}{n^s} \quad \text{and} \quad b(n) := \sum_{s=0}^{\infty} \frac{b_s}{n^s}, \tag{2.10}$$

where

$$a_0 = 2, \quad a_1 = 0, \quad a_2 = -4x - \frac{1}{4}, \cdots; \quad b_0 = 1, \quad b_1 = b_2 = \cdots = 0. \tag{2.11}$$

Thus, the characteristic equation associated with (2.8) is

$$\rho^2 + 2\rho + 1 = 0$$

and it has a double root $\rho = -1$. In addition, this root satisfies the auxiliary equation $a_1\rho + b_1 = 0$. In this case, according to Wong and Li [33], asymptotic solutions of (2.8) also involve roots of the indicial polynomial

$$q(\alpha) = \alpha(\alpha - 1) + 4x + \frac{1}{4},$$

which are

$$\alpha_1 = \frac{1}{2} + 2i\sqrt{x}, \quad \alpha_2 = \frac{1}{2} - 2i\sqrt{x}. \tag{2.12}$$

When $x > 0$, two linearly independent solutions of (2.8) are given by

$$\begin{aligned} y_j(n) &\sim \rho^n n^{\alpha_j} \sum_{s=0}^{\infty} \frac{c_{s,j}}{n^s} \\ &\sim (-1)^n n^{\frac{1}{2}} \exp\{\pm i2\sqrt{x}\log n\} \sum_{s=0}^{\infty} \frac{c_{s,j}}{n^s}, \quad j = 1, 2, \end{aligned} \tag{2.13}$$

as $n \to \infty$, where $c_{0,1} = c_{0,2} = 1$ and higher-order coefficients can be determined recursively; see [33]. Note that $\alpha_2 - \alpha_1 = 0$ if $x = 0$; and $\alpha_2 - \alpha_1 = m$ if $x = -m^2/16$ with $m = 1, 2, 3, \ldots$. In both cases, according to [33], Eq. (2.13) gives only one solution $y_1(n)$, a second solution is given by

$$y_2(n) \sim (-1)^n n^{\alpha_2} {\sum_{s=0}^{\infty}}' \frac{d_s}{n^s} + \bar{c}(\log n) y_1(n), \tag{2.14}$$

where the prime indicates that the term of order $\alpha_2 - \alpha_1$ is absent in the summation. The coefficients d_s and the constant $\bar{c}$ are determined by formal substitution, beginning with $\bar{c} = 1$ when $\alpha_2 - \alpha_1 = 0$, or with $d_0 = 1$ when $\alpha_2 - \alpha_1 = 1, 2, 3, \ldots$.

Now, $P_n(x)$ is a linear combination of the two linearly independent solutions, that is,

$$P_n(x) \sim C_1(x) y_1(n) + C_2(x) y_2(n), \quad n \to \infty \tag{2.15}$$

for some constants $C_1(x)$ and $C_2(x)$, which depend on x but not on n. According to Wilson [30, pp. 59–60], we have for large n and fixed $x > 0$

$$W_n(x) = 2\mathbf{C}_n\{\mathrm{Re}[n^{2i\sqrt{x}}\mathbf{A}(i\sqrt{x})] + \mathcal{O}\left(n^{-1}\right)\}, \tag{2.16}$$

where the constant $\mathbf{C}_n$ and the function $\mathbf{A}(z)$ are defined in (2.4) and (2.5), respectively.

Remark 1. Note that there is a typographical error in Wilson's original formula (last equation on [30, p. 59]); the negative sign there should be replaced by a positive sign. In view of (1.2) and (2.5), $|\mathbf{A}(i\sqrt{x})|^{-2} = |\mathbf{A}(-i\sqrt{x})|^{-2} = 2\sqrt{x}w(x)$ for $x > 0$. Thus, Eq. (2.16) actually gives the asymptotic behavior of $\sqrt{\sqrt{x}w(x)}W_n(x)$.

Wilson [30] also showed that for $\operatorname{Im} z > 0$ with $\mathbf{A}(-iz) \neq 0$,

$$W_n(z^2) \sim \mathbf{C}_n \mathbf{A}(-iz) n^{-2iz}, \quad n \to \infty. \tag{2.17}$$

Note from (1.4) and (1.13), we have

$$W_n(x) = (-1)^n (n+u-1)_n K_n P_n(x). \tag{2.18}$$

By Stirling's formula,

$$K_n \sim (4\pi)^2 2^{-u} n^{u-2} \left(\frac{n}{2e}\right)^{2n} \tag{2.19}$$

and

$$(n+u-1)_n = \frac{\Gamma(2n+u-1)}{\Gamma(n+u-1)} \sim e^{-n} n^n 2^{2n+u-3/2} \tag{2.20}$$

as $n \to \infty$. Thus,

$$(n+u-1)_n K_n \sim 2\sqrt{\pi}(2\pi)^{3/2} e^{-3n} n^{3n+u-2} = 2\sqrt{\pi} n^{-\frac{1}{2}} \mathbf{C}_n. \tag{2.21}$$

A combination of (2.13), (2.15), (2.16), (2.18), and (2.21) gives

$$C_1(x) = \frac{1}{2\sqrt{\pi}} \mathbf{A}(i\sqrt{x}), \quad C_2(x) = \frac{1}{2\sqrt{\pi}} \mathbf{A}(-i\sqrt{x}). \tag{2.22}$$

Hence,

$$W_n(x) \sim \mathbf{C}_n \left[\mathbf{A}(i\sqrt{x}) \exp\{i2\sqrt{x}\log n\} \sum_{s=0}^{\infty} \frac{c_{s,1}}{n^s} + \mathbf{A}(-i\sqrt{x}) \exp\{-i2\sqrt{x}\log n\} \sum_{s=0}^{\infty} \frac{c_{s,2}}{n^s} \right] \tag{2.23}$$

as $n \to \infty$, where $c_{0,1} = c_{0,2} = 1$.

Note that Eq. (2.23) also holds for complex x, provided that $\mathbf{A}(i\sqrt{x}) \neq 0$ and $\mathbf{A}(-i\sqrt{x}) \neq 0$; cf. Eq. (2.17). Note also that the functions $\mathbf{A}(\pm i\sqrt{x})$ do vanish due to the poles of the gamma functions appearing in the denominator of the function $\mathbf{A}(z)$ in (2.5). Finally, we observe that since the coefficients in the second-order difference equation (2.8) are all real, the coefficients $c_{s,1}$ and $c_{s,2}$ in (2.23) are also real. Furthermore, since $W_n(x)$ is real for $x > 0$, it follows that $c_{s,1} = c_{s,2}$ for $s = 0, 1, 2, \ldots$.

When $x = -m^2/16$, the roots of the indicial polynomial differ by a positive integer; see (2.12). Thus, (2.13) gives only one asymptotic solution $y_1(n)$; the other solution is given by (2.14) and contains a term of the form $\bar{c}(\log n) y_1(n)$. Since

$W_n(x)$ is continuous in x, we may take the limit $x \to -m^2/16$. From (2.14), (2.15), (2.18), (2.21), and [30, (2.5) and (2.7)], we obtain $\bar{c} = 0$ and

$$C_1(x) = \frac{1}{2\sqrt{\pi}}\mathbf{A}(i\sqrt{x}), \quad C_2(x) = \frac{1}{2\sqrt{\pi}}\mathbf{A}(-i\sqrt{x})$$

if m is an odd integer; and

$$C_1(x) = 0, \quad C_2(x) = \frac{1}{2\sqrt{\pi}}\mathbf{A}(-i\sqrt{x})$$

if m is an even integer. This implies that the asymptotic expansion (2.23) remains valid if $x = -(2k+1)^2/16$. When $x = -(2k)^2/16 = -k^2/4$, the asymptotic expansion becomes

$$W_n(x) \sim \mathbf{C}_n \mathbf{A}(-i\sqrt{x}) \exp\{-i2\sqrt{x}\log n\} \left[\sum_{s=0}^{\infty} \frac{\bar{c}_{s,2}}{n^s} + \bar{c}(\log n) \sum_{s=0}^{\infty} \frac{c_{s,1}}{n^{s+2k}}\right], \qquad (2.24)$$

where $\bar{c}_{0,2} = 1$, $\bar{c}_{2k,2} = 0$ and the other coefficients $\bar{c}_{s,2}$ and $\bar{c}$ can be determined recursively.

3. Scaled-x Asymptotic Approximations

In this section, we shall apply the method introduced in Wang and Wong [26] to derive large-n asymptotic approximations for $W_n(x)$ in both zero-free and oscillatory regions.

Theorem 2 (Scaled-x asymptotics). *Let γ_n and K_n be given by* (1.3) *and* (1.14), *respectively. The Wilson polynomials have the following asymptotic formulas as $n \to \infty$:*

(i) *Zero-free region*:

$$W_n(n^2 t) = \gamma_n \left(\frac{n}{2e}\right)^{2n} 2^{1-u} t^{\frac{3-2u}{4}} \left(\frac{1}{t-1}\right)^{\frac{1}{4}} \exp\left\{2n\sqrt{t}\arcsin\frac{1}{\sqrt{t}} + \left(n + \frac{u-1}{2}\right)\log(2t - 1 + 2\sqrt{t(t-1)})\right\}\left\{1 + \mathcal{O}\left(\frac{1}{n}\right)\right\} \qquad (3.1)$$

for t in any compact subset of $\mathbb{C}\backslash[0,1]$.

(ii) *Oscillatory region*:

$$W_n(n^2 t) = \gamma_n K_n (-1)^n \frac{(n^2 t)^{\frac{2-u}{2}}}{4\pi^2} e^{\pi\sqrt{n^2 t}} \left(\frac{1}{t(1-t)}\right)^{\frac{1}{4}} \times \left\{\cos\left[n\left(\arccos(1-2t) + \sqrt{t}\log\frac{2 - t + 2\sqrt{1-t}}{t}\right) + \frac{u-1}{2}\arccos(1-2t) + \frac{3-2u}{4}\pi\right] + \mathcal{O}\left(\frac{1}{n}\right)\right\} \qquad (3.2)$$

for t in any compact subset of $\mathbb{C}\backslash\{(-\infty, 0] \cup [1, \infty)\}$.

Note that the oscillatory region is a complex neighborhood of the oscillatory interval $(0,1)$. Moreover, Eqs. (3.1) and (3.2) are asymptotically equivalent for t bounded away from the real line.

3.1. *Zero-free region*

Set $w_0(x) = 1$ and let $w_k(x)$ be the ratio of two consecutive monic Wilson polynomials

$$w_k(x) = \frac{\pi_k(x)}{\pi_{k-1}(x)}. \tag{3.3}$$

Then,

$$\pi_n(x) = \prod_{k=1}^{n} w_k(x), \quad n = 1, 2, \dots. \tag{3.4}$$

Substituting this equation into the recurrence relation (1.5), we obtain

$$w_{k+1}(x) + A_k + C_k - a^2 - x + \frac{A_{k-1}C_k}{w_k(x)} = 0. \tag{3.5}$$

From (1.6) and (1.7), we have the large-n expansions of the coefficients

$$A_n + C_n - a^2 = \frac{1}{2}n^2 + \frac{a+b+c+d-1}{2}n + \mathcal{O}(1) \tag{3.6}$$

and

$$A_{n-1}C_n = \frac{1}{16}n^4 + \frac{a+b+c+d-2}{8}n^3 + \mathcal{O}\left(n^2\right). \tag{3.7}$$

Put

$$x := n^2 t, \quad k := ns, \tag{3.8}$$

and assume that the ratio $w_k(x)$ has the following asymptotic form:

$$w_k(x) = n^2 w(t,s)\left\{1 + \frac{\sigma(t,s)}{n} + O\left(\frac{1}{n^2}\right)\right\} \tag{3.9}$$

as $n \to \infty$. Taylor's expansion gives

$$\begin{aligned} w_{k+1}(x) &= n^2 w\left(t, s+\frac{1}{n}\right)\left\{1 + \frac{\sigma\left(t, s+\frac{1}{n}\right)}{n} + O\left(\frac{1}{n^2}\right)\right\} \\ &= n^2 w(t,s)\left\{1 + \frac{\sigma(t,s) + w_s(t,s)/w(t,s)}{n} + O\left(\frac{1}{n^2}\right)\right\} \end{aligned} \tag{3.10}$$

as $n \to \infty$. Substituting (3.9) and (3.10) into (3.5) and equating the coefficients of n^2 and n, we obtain

$$w + \frac{1}{2}s^2 - t + \frac{s^4}{16w} = 0 \tag{3.11}$$

and

$$w\sigma + w_s + \frac{u-1}{2}s + \frac{(2u-4)s^3 - s^4\sigma}{16w} = 0, \tag{3.12}$$

where $u = a+b+c+d$; cf. (2.1). Solving Eqs. (3.11) and (3.12) yields

$$w = \frac{\left(t - \frac{1}{2}s^2\right) + \sqrt{t(t-s^2)}}{2} \tag{3.13}$$

and

$$\sigma = \frac{16ww_s + 8(u-1)sw + (2u-4)s^3}{s^4 - 16w^2}. \tag{3.14}$$

By an induction argument, it is straightforward to show that the asymptotic formula (3.9) holds uniformly for $0 \leq s \leq 1$ and $t \in \mathbb{C}\backslash[0,1]$. To obtain the asymptotics of $\pi_n(x)$, we collect the contributions from all $w_k(x)$ in (3.4). By the trapezoidal rule, we have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\log w\left(t, \frac{k}{n}\right) &= \int_0^1 \log w(t,s)ds + \frac{\log w(t,1) - \log w(t,0)}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right)\\ &= -2 + \log\frac{2t-1+2\sqrt{t(t-1)}}{4} + 2\sqrt{t}\arcsin\frac{1}{\sqrt{t}}\\ &\quad + \frac{1}{2n}\log\frac{2t-1+2\sqrt{t(t-1)}}{4t} + \mathcal{O}\left(\frac{1}{n^2}\right)\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\sum_{k=1}^{n}\log\left[1 + \frac{\sigma\left(t, \frac{k}{n}\right)}{n}\right] &= \frac{1}{n}\sum_{k=1}^{n}\sigma\left(t, \frac{k}{n}\right) + \mathcal{O}\left(\frac{1}{n}\right)\\ &= \int_0^1 \sigma(t,s)ds + \mathcal{O}\left(\frac{1}{n}\right)\\ &= \frac{u-2}{2}\log\frac{2t-1+2\sqrt{t(t-1)}}{4t}\\ &\quad + \frac{1}{4}\log\frac{t}{t-1} + \mathcal{O}\left(\frac{1}{n}\right)\end{aligned} \tag{3.16}$$

as $n \to \infty$. In the evaluation of the integral in (3.15), we have first applied an integration by parts and then use the identity $w(t,s) = \frac{1}{4}(\sqrt{t} + \sqrt{t-s^2})^2$. To evaluate the integral in (3.16), we first calculate $s^2 - 4w$ and $s^2 + 4w$ from (3.13);

this leads to

$$s^4 - 16w^2 = -4\sqrt{t(t-s^2)}(\sqrt{t}+\sqrt{t-s^2})^2.$$

By writing $(2u-4)s^3$ as $(2u-2)s^3 - 2s^3$, we then obtain

$$\sigma(t,s) = -\frac{1}{2}[\log(s^4-16w^2)]' - \frac{2(u-1)s}{2\sqrt{t-s^2}(\sqrt{t}+\sqrt{t-s^2})}.$$

From (3.4) and (3.9), it follows that

$$\log \pi_n(x) = \sum_{k=1}^{n} \log w_k(x)$$

$$= 2n\log n + \sum_{k=1}^{n} \log w\left(t, \frac{k}{n}\right) + \frac{1}{n}\sum_{k=1}^{n} \log\left[1 + \frac{\sigma\left(t, \frac{k}{n}\right)}{n}\right] + \mathcal{O}\left(\frac{1}{n}\right).$$

The last equation, together with (3.15) and (3.16), yields

$$\begin{aligned}\pi_n(n^2t) = {} & \left(\frac{n}{2e}\right)^{2n}\left(\frac{2t-1+2\sqrt{t(t-1)}}{4t}\right)^{\frac{u-1}{2}}\left(\frac{t}{t-1}\right)^{\frac{1}{4}}\\ & \times \exp\left\{n\left[2\sqrt{t}\arcsin\frac{1}{\sqrt{t}} + \log(2t-1+2\sqrt{t(t-1)})\right]\right\}\\ & \times\left\{1+\mathcal{O}\left(\frac{1}{n}\right)\right\}, \end{aligned} \tag{3.17}$$

as $n \to \infty$, for t in any compact subset of $\mathbb{C}\backslash[0,1]$. A combination of (1.4), (2.2) and (3.17) gives (3.1).

3.2. *Oscillatory region*

To derive an asymptotic formula for the monic polynomial $\pi_n(x)$ in the oscillatory region, we present two methods. The first one is based on the observation that the asymptotic formula of $\pi_n(n^2t)$ for $t \in (0,1)$ is the sum of the two limits of the corresponding formula of $\pi_n(n^2t)$ for $t \in \mathbb{C}\backslash[0,1]$ as $\operatorname{Im} t \to \pm 0$; cf. [14, p. 175] and [23, §5a, p. 395]. The second method is more rigorous, and makes use of the results in [26], which were also used in Sec. 2 to obtain the fixed-x asymptotics of the Wilson polynomials $W_n(x)$; cf. (2.6) and (2.7).

Method I. For convenience, we put

$$\begin{aligned}\varphi_n(t) := {} & \left(\frac{1}{t-1}\right)^{\frac{1}{4}}\exp\left\{2n\sqrt{t}\arcsin\frac{1}{\sqrt{t}} + \left(n+\frac{u-1}{2}\right)\right.\\ & \left.\times\log(2t-1+2\sqrt{t(t-1)})\right\}\end{aligned} \tag{3.18}$$

and recall

$$\lim_{\varepsilon\to 0^+} \arcsin(x \pm i\varepsilon) = \frac{1}{2}\pi \pm i\log((x^2-1)^{\frac{1}{2}} + x), \quad x \in [1,\infty).$$

Coupling the last two equations gives

$$\begin{aligned}\varphi_n^{\pm}(t) &:= \lim_{\varepsilon\to 0^+} \varphi_n(t \pm i\varepsilon) \\ &= \exp\left\{n\sqrt{t}\left(\pi \mp i\log\frac{2-t+2\sqrt{1-t}}{t}\right)\right. \\ &\quad \left. + \left(n + \frac{u-1}{2}\right)\log(2t-1 \pm i2\sqrt{t}\sqrt{1-t}) - \frac{1}{4}\log(1-t) \mp i\frac{\pi}{4}\right\}. \qquad (3.19)\end{aligned}$$

Then, we have

$$\begin{aligned}\pi_n(n^2t) &\sim \left(\frac{n}{2e}\right)^{2n} 2^{1-u} t^{\frac{3-2u}{4}} [\varphi_n^+(t) + \varphi_n^-(t)] \\ &\sim \left(\frac{n}{2e}\right)^{2n} 2^{2-u} t^{\frac{3-2u}{4}} \frac{1}{(1-t)^{\frac{1}{4}}} \exp\{\pi n\sqrt{t}\} \cos\left\{n\sqrt{t}\log\frac{2-t+2\sqrt{1-t}}{t}\right. \\ &\quad \left. + \left(n + \frac{u-1}{2}\right)\arccos(1-2t) - \left(n + \frac{2u-3}{4}\right)\pi\right\}. \qquad (3.20)\end{aligned}$$

Here, we have made use of the identity [22, (4.23.22), p. 120]

$$\arccos z = \frac{1}{2}\pi + i\log(\sqrt{1-z^2} + iz), \quad z \in \mathbb{C}\backslash(-\infty,-1] \cup [1,\infty). \qquad (3.21)$$

Combining (1.4) and (3.20) gives (3.2).

Method II. We first recall Eq. (1.9), where $\widetilde{A}_n$ and $\widetilde{B}_n$ have the asymptotic expansions given in (1.16). Motivated by the argument given in [26], we assume

$$P_n(n^2t) \sim n^{\alpha}[\rho(t)]^n\{f(t)\cos(n\varphi(t)) + g(t)\sin(n\varphi(t))\}, \quad n \to \infty. \qquad (3.22)$$

(See also the presentation in Sec. 2.) Note that for fixed x, the value of t changes when n changes to $n \pm 1$. Let us put

$$x = n^2t := (n \pm 1)t_{\pm}. \qquad (3.23)$$

From (3.22), it follows that

$$\begin{aligned}P_{n\pm 1}(n^2t) &\sim (n \pm 1)^{\alpha}[\rho(t_{\pm})]^{n\pm 1}\{f(t_{\pm})\cos[(n \pm 1)\varphi(t_{\pm})] \\ &\quad + g(t_{\pm})\sin[(n \pm 1)\varphi(t_{\pm})]\}. \qquad (3.24)\end{aligned}$$

It is readily seen that

$$t_{\pm} = \left[1 \mp \frac{2}{n} + \frac{3}{n^2} + O\left(\frac{1}{n^3}\right)\right] t, \quad n \to \infty. \qquad (3.25)$$

By Taylor's expansion, we have

$$f(t_\pm) = f(t) \mp \frac{2}{n} t f'(t) + O\left(\frac{1}{n^2}\right), \quad g(t_\pm) = g(t) \mp \frac{2}{n} t g'(t) + O\left(\frac{1}{n^2}\right), \tag{3.26}$$

$$\begin{aligned}[\rho(t_\pm)]^{n\pm1} &= \left[\rho(t) \mp \frac{2}{n} t\rho'(t) + \frac{3t\rho'(t) + 2t^2\rho''(t)}{n^2} + O\left(\frac{1}{n^3}\right)\right]^{n\pm1} \\ &= [\rho(t)]^{n\pm1} \exp\left\{\mp 2t\frac{\rho'(t)}{\rho(t)} + \frac{t\rho'(t) + 2t^2\rho''(t)}{n\rho(t)} - \frac{2t^2\rho'(t)^2}{n\rho^2(t)}\right\} \\ &\quad \times \left\{1 + O\left(\frac{1}{n^2}\right)\right\}\end{aligned} \tag{3.27}$$

and

$$(n \pm 1)^\alpha = n^\alpha \left[1 \pm \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)\right] \tag{3.28}$$

as $n \to \infty$. Substituting (3.22) and (3.25)–(3.28) into (1.9), and making use of the result given in Lemma 1, we obtain by matching the leading-order terms on the two sides of the equation

$$(4t - 2)f = \rho e^{-2t\rho'/\rho}(f \cos u_0 + g \sin u_0) + \rho^{-1} e^{2t\rho'/\rho}(f \cos u_0 - g \sin u_0), \tag{3.29}$$

$$(4t - 2)g = \rho e^{-2t\rho'/\rho}(g \cos u_0 - f \sin u_0) + \rho^{-1} e^{2t\rho'/\rho}(g \cos u_0 + f \sin u_0), \tag{3.30}$$

where u_0 is related to the function $\varphi(t)$ in (3.22) via the formula

$$u_0 = \varphi(t) - 2t\varphi'(t). \tag{3.31}$$

Solving (3.29) and (3.30) yields

$$\rho(t) = -e^{c\sqrt{t}}, \tag{3.32}$$

$$\cos u_0 = 1 - 2t; \tag{3.33}$$

note that we have made our choice so that $u_0(0) = 0$. Coupling (3.31) and (3.33) gives

$$\varphi - 2t\varphi' = \arccos(1 - 2t), \tag{3.34}$$

which is a nonhomogeneous linear first-order ordinary differential equation. A particular solution of (3.34) is

$$\varphi(t) = \arccos(1 - 2t) + \sqrt{t} \log \frac{2 - t + 2\sqrt{1 - t}}{t}. \tag{3.35}$$

The problem of collecting terms of $O(\frac{1}{n})$ in the expansion on the left-hand side of (1.9) with $x = n^2 t$ becomes messier. However, with the exact formula in (3.32), we do not need to use (3.27) and, instead, we use the result

$$[\rho(t)]^n = (-1)^n e^{c\sqrt{n^2 t}} = (-1)^n e^{c\sqrt{(n\pm1)^2 t_\pm}} = -[\rho(t_\pm)]^{n\pm1}. \tag{3.36}$$

As a consequence, calculation simplifies tremendously, and the terms of order $O(\frac{1}{n})$ on the left-hand side of (1.9) with $x = n^2 t$ can be grouped together as

$$C\cos[n\varphi] + D\sin[n\varphi], \tag{3.37}$$

where

$$\begin{aligned} C &= +\alpha\{fG_0 + gH_0\} - 2t(f'G_0 + g'H_0) + (fG_1 + gH_1) \\ &\quad - \alpha\{fG_0 - gH_0\} + 2t(f'G_0 - g'H_0) + (-fG_1 + gH_1) \\ &= 2\alpha g H_0 - 2tg'H_0 + 2gH_1 \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} D &= +\alpha\{-fH_0 + gG_0\} - 2t(-f'H_0 + g'G_0) + (-fH_1 + gG_1) \\ &\quad - \alpha\{fH_0 + gG_0\} + 2t(f'H_0 + g'G_0) + (-fH_1 - gG_1) \\ &= -2\alpha f H_0 + 2tf'H_0 - 2fH_1. \end{aligned} \tag{3.39}$$

Note that replacing g by $-f$ in C, we obtain D. In deriving (3.37)–(3.39), we have again made use of Lemma 1 in Sec. 4, where we have also given the values of G_0, G_1, H_0, and H_1. Recall from (1.17) that $\alpha_1 = 4(1-u)$ and $\beta_1 = 0$. Since $H_0 = \sin u_0$ and $H_1 = u_1 \cos u_0$, upon comparing terms of order $O(\frac{1}{n})$ on two sides of Eq. (1.9), we conclude

$$2(u-1)tf = \alpha g \sin u_0 - tg' \sin u_0 + gu_1 \cos u_0, \tag{3.40}$$

$$2(1-u)tg = \alpha f \sin u_0 - tf' \sin u_0 + fu_1 \cos u_0; \tag{3.41}$$

here we have taken into account the minus sign in (3.36). Solving Eqs. (3.40) and (3.41), we have

$$\left(\frac{f}{g}\right)^2 + 1 = \frac{2}{u-1}\frac{d}{dt}\left(\frac{f}{g}\right) \tag{3.42}$$

and

$$\frac{(fg)'}{fg} = -\frac{(\sin u_0)'}{\sin u_0} + \frac{\alpha}{t} + \left(\frac{g}{f} - \frac{f}{g}\right)\frac{u-1}{\sin u_0}, \tag{3.43}$$

where "$'$" denotes the derivative with respect to t and we have used (3.33) and the expressions of u_0 and u_1 in (4.13) below. Equation (3.42) is a nonlinear first-order differential equation, which can be solved explicitly. Its solution is

$$\frac{f}{g} = \tan\left[\frac{u-1}{2}u_0 + D\right], \tag{3.44}$$

where D is a constant. To solve the equation in (3.43), we note that $\cot\theta - \tan\theta = 2\cot(2\theta)$. Hence, from (3.44), it follows that

$$\frac{g}{f} - \frac{f}{g} = 2\cot[(u-1)u_0 + 2D]$$
$$= \frac{\{\sin[(u-1)u_0 + 2D]\}'}{\sin[(u-1)u_0 + 2D]} \frac{2}{(u-1)u_0'}.$$

Since $u_0' = 2/(\sin u_0)$ by (3.33), each term in (3.43) can be integrated and we obtain

$$fg = Ct^\alpha \frac{1}{\sin u_0} \sin[(u-1)u_0 + 2D], \tag{3.45}$$

where C is also a constant. Multiplying the two equations in (3.44) and (3.45), and taking a square root, gives

$$f(t) = (2C)^{\frac{1}{2}} \frac{t^{\alpha/2}}{(\sin u_0)^{1/2}} \sin\left[\frac{u-1}{2} u_0 + D\right]. \tag{3.46}$$

Coupling (3.44) and (3.46), we have

$$g(t) = (2C)^{\frac{1}{2}} \frac{t^{\alpha/2}}{(\sin u_0)^{1/2}} \cos\left[\frac{u-1}{2} u_0 + D\right]. \tag{3.47}$$

Without loss of generality, we may replace $C^{1/2}$ by C in the above two equations. Since $u_0 = \arccos(1-2t)$ and $\sin u_0 = 2[t(1-t)]^{1/2}$ by (3.33), this leads to

$$f(t) = C \frac{t^{\alpha/2}}{[t(1-t)]^{1/4}} \sin\left[\frac{u-1}{2} \arccos(1-2t) + D\right] \tag{3.48}$$

and

$$g(t) = C \frac{t^{\alpha/2}}{[t(1-t)]^{1/4}} \cos\left[\frac{u-1}{2} \arccos(1-2t) + D\right]. \tag{3.49}$$

Substituting (3.32), (3.35), (3.48), and (3.49) into (3.22) yields

$$\begin{aligned} P_n(n^2 t) &\sim Cn^\alpha (-1)^n \frac{e^{nc\sqrt{t}} t^{\alpha/2}}{[t(1-t)]^{1/4}} \left\{ \sin\left[\frac{u-1}{2} \arccos(1-2t) + D\right] \cos(n\varphi) \right. \\ &\quad \left. + \cos\left[\frac{u-1}{2} \arccos(1-2t) + D\right] \sin(n\varphi) \right\} \\ &\sim Cn^\alpha (-1)^n \frac{e^{nc\sqrt{t}} t^{\alpha/2}}{[t(1-t)]^{1/4}} \left\{ \cos\left[n\varphi + \frac{u-1}{2} \arccos(1-2t) + D - \frac{\pi}{2}\right] \right\} \end{aligned} \tag{3.50}$$

as $n \to \infty$ for $0 < t < 1$.

The idea of Wang and Wong [26] in determining the constants in the above formulas is based on the observation that Eq. (3.50) holds, in fact, for t in any compact subset of $\mathbb{C}\backslash\{(-\infty,0]\cup[1,\infty)\}$. Thus, it has a common region of validity in the complex t-plane with the formula in (3.17). To this end, we recall formulas in (1.13) and (2.19), so that (3.17) can rewritten as

$$P_n(n^2t) \sim \frac{2^u n^{2-u}}{(4\pi)^2}\left(\frac{1}{4t}\right)^{\frac{u-1}{2}}\left(\frac{t}{t-1}\right)^{\frac{1}{4}}\exp\left\{2n\sqrt{t}\arcsin\frac{1}{\sqrt{t}} + \left(n+\frac{u-1}{2}\right)\log(2t-1+2\sqrt{t(t-1)})\right\} \tag{3.51}$$

as $n\to\infty$ for $t\in\mathbb{C}\backslash[0,1]$.

To compare (3.50) with (3.51), we express the inverse trigonometric functions in terms of the logarithm function:

$$2\arcsin\frac{1}{\sqrt{t}} = \begin{cases} \pi - i\log\left\{\dfrac{2-t+2\sqrt{1-t}}{t}\right\}, & \operatorname{Im} t>0,\\[2ex] \pi + i\log\left\{\dfrac{2-t+2\sqrt{1-t}}{t}\right\}, & \operatorname{Im} t<0,\end{cases} \tag{3.52}$$

$$\arccos(1-2t) = \begin{cases} \pi + i\log\{2t-1+2\sqrt{t(t-1)}\}, & \operatorname{Im} t>0,\\ \pi - i\log\{2t-1+2\sqrt{t(t-1)}\}, & \operatorname{Im} t<0;\end{cases} \tag{3.53}$$

see [22, (4.23.20) and (4.23.25)]. Note that $\operatorname{Im}(\arccos(1-2t))>0$ and $\operatorname{Im}\varphi>0$ when $\operatorname{Im} t>0$. Comparing powers of n in (3.50) and (3.51), it is easily seen that we have $\alpha = 2-u$. Replacing the arcsine function in (3.51) by the logarithm function in (3.52), one immediately finds the exponential function $e^{n\pi\sqrt{t}}$ in (3.51); thus $c=\pi$. To calculate the constant C in (3.50), we let θ denote the argument of the cosine function in (3.50). Then, from (3.51), we have

$$\frac{C}{2} = \frac{2^u}{(4\pi)^2}\left(\frac{1}{4}\right)^{\frac{u-1}{2}} \quad \text{and} \quad C = \frac{1}{4\pi^2},$$

where the factor $\frac{1}{2}$ on the left-hand side comes from the formula $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$. Since $\operatorname{Re}(i\theta)\to-\infty$ as $n\to\infty$, we have $\cos\theta\approx\frac{1}{2}e^{-i\theta}$. From (3.35) and (3.53), we obtain

$$\theta = \left(n+\frac{u-1}{2}\right)[\pi + i\log(2t-1+2\sqrt{t(t-1)})] + n\sqrt{t}\log\frac{2-t+2\sqrt{1-t}}{t} + D - \frac{\pi}{2}. \tag{3.54}$$

Let ω denote the quantity inside the curly brackets of the exponential function in (3.51). From (3.52), it follows that

$$\omega = n\sqrt{t}\left[\pi - i\log\left(\frac{2-t+2\sqrt{2-t}}{t}\right)\right] + \left(n+\frac{u-1}{2}\right)\log(2t-1+2\sqrt{t(t-1)}). \tag{3.55}$$

Matching the exponential terms in (3.50) and (3.51) gives

$$e^{nc\sqrt{t}-i\theta+n\pi i} = e^{\omega-\frac{\pi}{4}i}, \tag{3.56}$$

where the exponent $-\frac{\pi}{4}i$ on the right-hand side comes from changing $(1-t)^{-1/4}$ to $(t-1)^{-1/4}$. Since $c=\pi$, it follows from (3.56)

$$\left(n+\frac{u-1}{2}\right)\pi - n\pi + D - \frac{\pi}{2} = \frac{\pi}{4}.$$

Substituting the values

$$\alpha = 2-u, \quad c=\pi, \quad C=\frac{1}{4\pi^2}, \quad D=\frac{5-2u}{4}\pi \tag{3.57}$$

into (3.50) leads to

$$P_n(n^2t) \sim (-1)^n\frac{n^{2-u}t^{\frac{2-u}{2}}}{4\pi^2}\frac{e^{n\pi\sqrt{t}}}{[t(1-t)]^{1/4}} \times\cos\left[n\varphi+\frac{u-1}{2}\arccos(1-2t)+\frac{3-2u}{4}\pi\right], \tag{3.58}$$

where φ is given in (3.35). To compare this result with the one in (3.20) obtained earlier by using Method I, we note that on account of (1.13) and (2.19), Eq. (3.20) can be written in the form

$$P_n(n^2t) \sim \frac{n^{2-u}}{4\pi^2}e^{\pi n\sqrt{t}}\frac{t^{(2-u)/2}}{[t(1-t)]^{1/4}} \times\cos\left\{n\left[\sqrt{t}\log\frac{2-t+2\sqrt{1-t}}{t}+\arccos(1-2t)\right] + \left(\frac{u-1}{2}\right)\arccos(1-2t)+\frac{3-2u}{4}\pi - n\pi\right\}. \tag{3.59}$$

Since the quantity inside the square bracket is equal to φ by (3.35) and since $(-1)^n\cos\theta = \cos(\theta-n\pi)$, the last equation agrees with (3.58).

4. Transition Point at the Origin: An Exceptional Case

Before proceeding further with the Wilson polynomials, we need to consider the exceptional case mentioned in Sec. 1. Thus, we return to the second-order linear difference equation (1.9), where the coefficients have the general asymptotic expansions given in (1.10) with $\theta = 2$, $\alpha_0 > 0$ and $\beta_0 = -2$. In the neighborhood of the transition point $t_1 = 0$, we have the following uniform asymptotic result.

Theorem 3. *Assume $\theta = 2$, $\alpha_0 > 0$ and $\beta_0 = -2$. Let $N = n + \tau_0$ with $\tau_0 = -\alpha_1/(2\alpha_0)$. As $n \to \infty$, Eq.* (1.9) *has two linearly independent asymptotic solutions*

$$\mathcal{P}_n(N^2T) \sim (-1)^n \left[\frac{1}{T(1 - T\alpha_0/4)}\right]^{\frac{1}{4}} \cos\left(N\eta(T\alpha_0/4)\right) \sum_{s=0}^{\infty} \frac{C_s(\eta)}{N^s} \tag{4.1}$$

and

$$\mathcal{Q}_n(N^2T) \sim (-1)^n \left[\frac{1}{T(1 - T\alpha_0/4)}\right]^{\frac{1}{4}} \sin(N\eta(T\alpha_0/4)) \sum_{s=0}^{\infty} \frac{D_s(\eta)}{N^s} \tag{4.2}$$

for $-\infty < T < t_2 - \delta$, where $t_2 = 4/\alpha_0$ is the positive transition point, $\delta > 0$ and $\eta(T)$ is given by

$$\eta(T) := \arccos(1 - 2T) + \sqrt{T} \log \frac{2 - T + 2\sqrt{1 - T}}{T}. \tag{4.3}$$

Here, $C_0(\eta) = D_0(\eta) = 1$, while $C_s(\eta)$ and $D_s(\eta)$ with $s > 0$ can be determined recursively.

Note that the coefficients $C_s(\eta)$ in (4.1) should not be confused with the constants C_n in (1.7).

Remark 2. To see how the assumptions on the coefficients α_0 and β_0 in the above theorem can be removed, we refer to [4, Remark 4].

Upon a multiplication of $(-1)^n$ on $P_n(x)$, we consider the three-term recurrence relation given in (1.9)–(1.10) with $\theta = 2$ and the first two coefficients α_0 and β_0 satisfying

$$\alpha_0 < 0 \quad \text{and} \quad \beta_0 = 2. \tag{4.4}$$

From (1.12), one readily sees that the second condition in (4.4) implies that a transition point occurs at the origin, and that the first condition in (4.4) implies that the other transition point is to the right of the origin. Let

$$\tau_0 = -\frac{\alpha_1}{2\alpha_0} \quad \text{and} \quad N = n + \tau_0, \tag{4.5}$$

and re-expand the recurrence coefficients $\widetilde{A}_n$ and $\widetilde{B}_n$ in (1.10) in terms of N:

$$\widetilde{A}_n = \frac{1}{N^2} \sum_{s=0}^{\infty} \frac{\alpha'_s}{N^s}, \quad \widetilde{B}_n = \sum_{s=0}^{\infty} \frac{\beta'_s}{N^s}. \tag{4.6}$$

The first few coefficients are given by

$$\alpha_0' = \alpha_0, \quad \alpha_1' = 0, \quad \beta_0' = \beta_0 = 2, \quad \beta_1' = \beta_1, \quad \beta_2' = \beta_2 - \frac{\alpha_1}{2\alpha_0}\beta_1. \tag{4.7}$$

Moreover, we assume that $\beta_1 = 0$, so that

$$\alpha_1' t_1 + \beta_1' = \beta_1' = \beta_1 = 0. \tag{4.8}$$

Put $x = N^2 T$. Note that the variable x in the recurrence relation (1.9) is fixed. We write, as in (3.23),

$$x = N^2 T = (N \pm 1)^2 T_\pm. \tag{4.9}$$

(The values of $T_\pm$ here, of course, differ from $t_\pm$ in (3.23).) We need the following lemma in the proof of our Theorem 3.

Lemma 1. *Let $\eta(T)$ be a C^∞-function in $(0, \infty)$. We have*

$$\cos\{(N \pm 1)\eta(T_\pm)\} = \cos(N\eta) G_\pm\left(\eta, \frac{1}{N}\right) - \sin(N\eta) H_\pm\left(\eta, \frac{1}{N}\right), \tag{4.10}$$

$$\sin\{(N \pm 1)\eta(T_\pm)\} = \sin(N\eta) G_\pm\left(\eta, \frac{1}{N}\right) + \cos(N\eta) H_\pm\left(\eta, \frac{1}{N}\right), \tag{4.11}$$

as $N \to \infty$, where

$$G_\pm\left(\eta, \frac{1}{N}\right) = \sum_{s=0}^{\infty} (\pm 1)^s \frac{G_s(\eta)}{N^s}, \quad H_\pm\left(\eta, \frac{1}{N}\right) = \sum_{s=0}^{\infty} (\pm 1)^{s+1} \frac{H_s(\eta)}{N^s}.$$

Proof. Let

$$(N \pm 1)\eta(T_\pm) - N\eta(T) := u_\pm = \sum_{s=0}^{\infty} \frac{u_s^\pm}{N^s}. \tag{4.12}$$

From the definition of $T_\pm$, we have

$$T_\pm = \left(1 \pm \frac{1}{N}\right)^{-2} T = T \sum_{s=0}^{\infty} (\mp 1)^s \frac{s+1}{N^s}.$$

Taylor's expansion gives

$$\eta(T_\pm) = \eta(T) + \sum_{k=1}^{\infty} \frac{\eta^{(k)}(T)}{k!} \left[T \sum_{s=1}^{\infty} (\mp 1)^s \frac{s+1}{N^s}\right]^k,$$

which yields

$$u_\pm = \sum_{s=0}^{\infty} (\pm 1)^{s+1} \frac{u_s}{N^s},$$

where

$$u_0 = \eta - 2T\eta', \quad u_1 = T\eta' + 2T^2\eta'', \quad u_2 = -T\eta' - 4T^2\eta'' - \frac{4}{3}T^3\eta'''. \tag{4.13}$$

From (4.12), it follows

$$\cos((N \pm 1)\eta) = \cos(N\eta + u_\pm) = \cos(N\eta)\cos(u_\pm) - \sin(N\eta)\sin(u_\pm).$$

We use the notations

$$G_\pm\left(\eta, \frac{1}{N}\right) := \cos(u_\pm) = \sum_{s=0}^{\infty}(\pm 1)^s \frac{G_s(\eta)}{N^s},$$

$$H_\pm\left(\eta, \frac{1}{N}\right) := \sin(u_\pm) = \sum_{s=0}^{\infty}(\pm 1)^{s+1} \frac{H_s(\eta)}{N^s},$$

where

$$G_0 = \cos u_0, \quad G_1 = -u_1 \sin u_0, \quad H_0 = \sin u_0, \quad H_1 = u_1 \cos u_0. \tag{4.14}$$

Expansion (4.11) is obtained in a similar manner. □

We try a formal series solution of (1.9) in the form

$$P_n(x) \sim \chi(N\eta)\sum_{s=0}^{\infty}\frac{C_s(\eta)}{N^s} + \chi'(N\eta)\sum_{s=0}^{\infty}\frac{D_s(\eta)}{N^s}, \tag{4.15}$$

where $x = N^2T$, η is a function of T and $\chi(\xi) = \cos\xi$ or $\sin\xi$. For convenience, we set

$$C\left(\eta, \frac{1}{N}\right) = \sum_{s=0}^{\infty}\frac{C_s(\eta)}{N^s}, \quad D\left(\eta, \frac{1}{N}\right) = \sum_{s=0}^{\infty}\frac{D_s(\eta)}{N^s} \tag{4.16}$$

and

$$\Psi\left(T, \frac{1}{N}\right) := \widetilde{A}_n x + \widetilde{B}_n = \sum_{s=0}^{\infty}\frac{\alpha'_s T + \beta'_s}{N^s}; \tag{4.17}$$

cf. Eqs. (1.9)–(1.10). From Lemma 1, we have

$$\begin{aligned} P_{n\pm1}(x) = \chi(N\eta)\Bigg\{&G_\pm\left(\eta, \frac{1}{N}\right) C\left(\eta(T_\pm), \frac{1}{N \pm 1}\right) \\ &- H_\pm\left(\eta, \frac{1}{N}\right) D\left(\eta(T_\pm), \frac{1}{N \pm 1}\right)\Bigg\} \\ &+ \chi'(N\eta)\Bigg\{G_\pm\left(\eta, \frac{1}{N}\right) D\left(\eta(T_\pm), \frac{1}{N \pm 1}\right) \\ &+ H_\pm\left(\eta, \frac{1}{N}\right) C\left(\eta(T_\pm), \frac{1}{N \pm 1}\right)\Bigg\}. \end{aligned} \tag{4.18}$$

Substituting (4.15)–(4.18) into (1.9), and matching the coefficients of χ and χ', we obtain

$$G_+\left(\eta, \frac{1}{N}\right) C\left(\eta(T_+), \frac{1}{N+1}\right) + G_-\left(\eta, \frac{1}{N}\right) C\left(\eta(T_-), \frac{1}{N-1}\right)$$
$$- H_+\left(\eta, \frac{1}{N}\right) D\left(\eta(T_+), \frac{1}{N+1}\right) - H_-\left(\eta, \frac{1}{N}\right) D\left(\eta(T_-), \frac{1}{N-1}\right)$$
$$- \Psi\left(T, \frac{1}{N}\right) C\left(\eta, \frac{1}{N}\right) = 0 \tag{4.19}$$

and

$$G_+\left(\eta, \frac{1}{N}\right) D\left(\eta(T_+), \frac{1}{N+1}\right) + G_-\left(\eta, \frac{1}{N}\right) D\left(\eta(T_-), \frac{1}{N-1}\right)$$
$$+ H_+\left(\eta, \frac{1}{N}\right) C\left(\eta(T_+), \frac{1}{N+1}\right) + H_-\left(\eta, \frac{1}{N}\right) C\left(\eta(T_-), \frac{1}{N-1}\right)$$
$$- \Psi\left(T, \frac{1}{N}\right) D\left(\eta, \frac{1}{N}\right) = 0. \tag{4.20}$$

Letting $N \to \infty$ in the last two equations, we have

$$G_0(\eta) = \frac{\alpha_0' T + \beta_0'}{2}, \tag{4.21}$$

which, together with (4.14) and (4.13), yields

$$\eta(T) - 2T\eta'(T) = \arccos\left(\frac{\alpha_0' T + \beta_0'}{2}\right), \tag{4.22}$$

where the arccos function is analytically continued to $\mathbb{C}\backslash(-\infty, 0] \cup [t_2, \infty)$ with $t_2 = -4/\alpha_0 > 0$ as determined by (1.12) and (4.4). A particular solution of Eq. (4.22) can be found in exactly the same manner as Eq. (3.34), and we have

$$\eta(T) = \arccos\left(1 + \frac{\alpha_0'}{2}T\right) + \frac{\sqrt{-\alpha_0'}}{2}\sqrt{T}\log\left\{\frac{-4}{\alpha_0' T}\left[2 + \frac{\alpha_0'}{4}T + 2\sqrt{1 + \frac{\alpha_0'}{4}T}\right]\right\}. \tag{4.23}$$

In the case of Wilson polynomials, $\alpha_0' = -4$ and the above equation reduces to (3.35).

Equating the coefficients of like powers of $1/N$ in (4.19)–(4.20), we obtain

$$\sum_{s \le p, s \text{ even}} \sum_{i+m \le s} G_i(\eta) \binom{-p+s}{s-i-m} \sum_{l=0}^{m} \frac{D^l C_{p-s}(\eta)}{l!} T^l \gamma_{l,m}$$
$$- \sum_{s \le p, s \text{ odd}} \sum_{i+m \le s} H_i(\eta) \binom{-p+s}{s-i-m} \sum_{l=0}^{m} \frac{D^l D_{p-s}(\eta)}{l!} T^l \gamma_{l,m}$$
$$- \frac{1}{2} \sum_{s \le p} (\alpha_s' T + \beta_s') C_{p-s}(\eta) = 0 \tag{4.24}$$

and

$$\sum_{s\le p, s \text{ even}} \sum_{i+m\le s} G_i(\eta) \binom{-p+s}{s-i-m} \sum_{l=0}^{m} \frac{D^l D_{p-s}(\eta)}{l!} T^l \gamma_{l,m}$$
$$+ \sum_{s\le p, s \text{ odd}} \sum_{i+m\le s} H_i(\eta) \binom{-p+s}{s-i-m} \sum_{l=0}^{m} \frac{D^l C_{p-s}(\eta)}{l!} T^l \gamma_{l,m}$$
$$- \frac{1}{2} \sum_{s\le p} (\alpha_s' T + \beta_s') D_{p-s}(\eta) = 0, \tag{4.25}$$

where D^j denotes the jth derivative with respect to T, i.e.

$$D^j A_l(\eta) = \frac{d^j}{dT^j} A_l(\eta(T)), \quad l = 0, 1, 2, \ldots, \tag{4.26}$$

and $\gamma_{l,m}$ with $l \ge m$ is defined by

$$\sum_{m=l}^{\infty} \frac{\gamma_{l,m}}{N^m} := \left[\left(1 + \frac{1}{N}\right)^{-2} - 1 \right]^l . \tag{4.27}$$

For convenience, we put

$$f_{p-1}(T) := \sum_{2\le s\le p} \frac{\alpha_s' T + \beta_s'}{2} C_{p-s}(\eta)$$
$$- \sum_{\substack{2\le s\le p \\ s \text{ even}}} \left[\sum_{i+m\le s} \binom{-p+s}{s-i-m} G_i(\eta) \sum_{l=0}^{m} \frac{D^l C_{p-s}(\eta)}{l!} \gamma_{l,m} T^l \right]$$
$$+ \sum_{\substack{2\le s\le p \\ s \text{ odd}}} \left[\sum_{i+m\le s} \binom{-p+s}{s-i-m} H_i(\eta) \sum_{l=0}^{m} \frac{D^l D_{p-s}(\eta)}{l!} \gamma_{l,m} T^l \right] \tag{4.28}$$

and

$$g_{p-1}(T) := \sum_{2\le s\le p} \frac{\alpha_s' T + \beta_s'}{2} D_{p-s}(\eta)$$
$$- \sum_{\substack{2\le s\le p \\ s \text{ even}}} \left[\sum_{i+m\le s} \binom{-p+s}{s-i-m} G_i(\eta) \sum_{l=0}^{m} \frac{D^l D_{p-s}(\eta)}{l!} \gamma_{l,m} T^l \right]$$
$$- \sum_{\substack{2\le s\le p \\ s \text{ odd}}} \left[\sum_{i+m\le s} \binom{-p+s}{s-i-m} H_i(\eta) \sum_{l=0}^{m} \frac{D^l C_{p-s}(\eta)}{l!} \gamma_{l,m} T^l \right] \tag{4.29}$$

for $p \geq 1$. Clearly, $f_0(T) = g_0(T) = 0$. Note that $\gamma_{0,0} = 1$, $\gamma_{0,m} = 0$, $m = 1, 2, \ldots$, and $\gamma_{1,0} = 0, \gamma_{1,1} = -2$. By (4.28)–(4.29) and (4.7)–(4.8), Eqs. (4.24) and (4.25) can be rewritten as

$$[(1-p)H_0(\eta) + H_1(\eta)]D_{p-1}(\eta) - 2TH_0(\eta)\frac{d}{dT}D_{p-1}(\eta) = f_{p-1}(T) \tag{4.30}$$

and

$$[(1-p)H_0(\eta) + H_1(\eta)]C_{p-1}(\eta) - 2TH_0(\eta)\frac{d}{dT}C_{p-1}(\eta) = g_{p-1}(T), \tag{4.31}$$

where we have made use of (4.21) and the assumption that $\alpha_1' = \beta_1' = 0$; cf. (4.7) and (4.8). In view of (4.14) and (4.21), we have

$$H_0(\eta) = \sqrt{\frac{4 - (\alpha_0' T + \beta_0')^2}{4}}, \quad \frac{dH_0(\eta)}{dT} = -\frac{\alpha_0'}{2}\frac{G_0(\eta)}{H_0(\eta)}. \tag{4.32}$$

Differentiating (4.22) gives

$$-\eta'(T) - 2T\eta''(T) = -\frac{\alpha_0'}{2H_0(\eta)}. \tag{4.33}$$

A combination of (4.14), (4.13), (4.32), and (4.33) yields

$$H_1(\eta) = -T\frac{d}{dT}H_0(\eta). \tag{4.34}$$

Substituting (4.34) into (4.30) and (4.31), we obtain

$$\begin{cases} \dfrac{d}{dT}[T^{\frac{p-1}{2}}\Lambda(T)D_{p-1}] = -\dfrac{1}{2}T^{\frac{p-1}{2}-1}\Lambda(T)^{-1}f_{p-1}(T), \\ \dfrac{d}{dT}[T^{\frac{p-1}{2}}\Lambda(T)C_{p-1}] = -\dfrac{1}{2}T^{\frac{p-1}{2}-1}\Lambda(T)^{-1}g_{p-1}(T), \end{cases} \tag{4.35}$$

where

$$\Lambda(T) := [H_0(\eta)]^{\frac{1}{2}}. \tag{4.36}$$

We take $p = 1$ in (4.35). Since $f_0 = g_0 = 0$, $\Lambda C_0(\eta)$ and $\Lambda D_0(\eta)$ are constants. Without loss of generality, we may set

$$\Lambda C_0(\eta) = 1, \quad \Lambda D_0(\eta) = 1. \tag{4.37}$$

For $p > 1$, (4.35) gives

$$T^{\frac{p-1}{2}}\Lambda(T)D_{p-1}(\eta) = -\frac{1}{2}\int_0^T s^{\frac{p-1}{2}-1}\Lambda(s)^{-1}f_{p-1}(s)ds \tag{4.38}$$

and

$$T^{\frac{p-1}{2}}\Lambda(T)C_{p-1}(\eta) = -\frac{1}{2}\int_0^T s^{\frac{p-1}{2}-1}\Lambda(s)^{-1}g_{p-1}(s)ds \tag{4.39}$$

for $T < t_2 - \delta$. Note that for $p \geq 2$, the above two integrals have a singularity of the order $O(s^{-3/4})$ at the origin. From (4.38) and (4.39), it follows that for each $p > 1$, $C_p(\eta)$ and $D_p(\eta)$ can be determined successively from their predecessors $C_0(\eta), D_0(\eta), \ldots, C_{p-1}(\eta)$ and $D_{p-1}(\eta)$. By using a similar argument as in [4, 27, 28], we can further prove that the formal series solution $P_n(x)$ in (4.15) (with χ being either the cosine or sine function) is indeed an asymptotic solution. This proves Theorem 3.

5. Uniform Asymptotic Approximations

Theorem 4 (Uniform asymptotics). *Let* $N = n + \frac{u-1}{2}$, *where* u *is defined in* (2.1). *The Wilson polynomials have the following uniform asymptotic expansions as* $n \to \infty$:

(i) *near the transition point* $t_2 = 1$,

$$[w(N^2T)]^{\frac{1}{2}} W_n(N^2T) \sim \gamma_n K_n N^{\frac{1}{6}} \left(\frac{\zeta}{4T(T-1)}\right)^{\frac{1}{4}} \times \left[\mathrm{Ai}(N^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} + N^{-\frac{1}{3}} \mathrm{Ai}'(N^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s}\right], \tag{5.1}$$

where $0 < \delta \leq T < \infty$, w *is the weight function given in* (1.2), *and* $\zeta(T)$ *is given by*

$$\frac{2}{3}[\zeta(T)]^{\frac{3}{2}} = 2\sqrt{T}\arctan(\sqrt{T-1}) - \log(2T - 1 + 2\sqrt{T(T-1)}), \quad T \geq t_1 = 1, \tag{5.2}$$

and

$$\frac{2}{3}[-\zeta(T)]^{\frac{3}{2}} = \arccos(2T-1) - 2\sqrt{T}\mathrm{arctanh}\,(\sqrt{1-T}), \quad 0 < T < 1. \tag{5.3}$$

(ii) *near the transition point* $t_1 = 0$,

$$W_n(N^2T) \sim \frac{(-1)^n \gamma_n K_n \sqrt{N}}{(1-T)^{1/4}\sqrt{\pi}} \left\{ \mathrm{Re}[(2e/\sqrt{x})^{2i\sqrt{x}} \mathbf{A}(-i\sqrt{x})] \times \cos(N\eta(T)) \sum_{s=0}^{\infty} \frac{C_s(\eta)}{N^s} + \mathrm{Im}[(2e/\sqrt{x})^{2i\sqrt{x}} \mathbf{A}(-i\sqrt{x})] \times \sin(N\eta(T)) \sum_{s=0}^{\infty} \frac{D_s(\eta)}{N^s} \right\}, \tag{5.4}$$

where $-\infty < T \leq 1 - \delta < 1$, $x = N^2T$ *and* $\eta(T)$ *is given by* (4.3). *Here,* $C_0(\eta) = D_0(\eta) = 1$, *while* $C_s(\eta)$ *and* $D_s(\eta)$ *with* $s > 0$ *can be determined recursively.*

Note that the coefficients $A_s(\zeta)$ in (5.1) should not be confused with the constants A_n in (1.6).

Remark 3. Masson [19] introduced the associated Wilson polynomials, which are closely related to the Wilson polynomials. The only difference is that the associated Wilson polynomials are generated by a recurrence relation which are obtained from the recurrence relation (1.5) by simply replacing n by $n+\varepsilon$, where ε denotes some constant. In a manner similar to what we are doing in this paper, one can obtain corresponding asymptotic approximations for the associated Wilson polynomials. To save space, we do not list these results here.

5.1. *Asymptotic approximations around the transition point* $t_1 = 0$

To derive an asymptotic formula for $W_n(n^2 t)$, which holds uniformly when t lies in a neighborhood of the origin (namely, $|t| < \delta$ with δ small and independent of n), we first note from (1.4) that (3.20) is equivalent to

$$W_n(n^2 t) \sim (-1)^n \left(\frac{n}{e}\right)^{3n} 2^{\frac{1}{2}} t^{\frac{3}{4}-\frac{u}{2}} (1-t)^{-\frac{1}{4}} e^{n\pi\sqrt{t}} \times \mathrm{Re}\left[\exp\left\{ i(2n+u-1)\arccos\sqrt{t} + i2n\sqrt{t}\log\frac{\sqrt{t}}{1+\sqrt{1-t}} - i\frac{\pi}{4}\right\}\right]. \tag{5.5}$$

To see this, we let $\sqrt{t} = \cos\theta$. Using trigonometric identities, it is readily seen that $1-2t = -\cos 2\theta$ and $\arccos(1-2t) = \pi - 2\arccos\sqrt{t}$. Also, it is evident that

$$2\log\frac{\sqrt{t}}{1+\sqrt{1-t}} = \log\frac{t}{2-t+2\sqrt{1-t}}.$$

On the other hand, we have from (2.16)

$$W_n(x) \sim \left(\frac{2\pi}{n}\right)^{3/2} \left(\frac{n}{e}\right)^{3n} n^u \mathrm{Re}[n^{2i\sqrt{x}} \mathbf{A}(i\sqrt{x})], \tag{5.6}$$

where $\mathbf{A}(x)$ is given in (2.5). By Stirling's formula, the behavior of $\mathbf{A}(i\sqrt{x})$ is given by

$$\mathbf{A}(i\sqrt{x}) \sim 2^{-2}\pi^{-3/2} x^{3/4-u/2} \exp\{\pi\sqrt{x} + i\pi(3/4-u/2) + 2i\sqrt{x}[\log(2/\sqrt{x})+1]\} \tag{5.7}$$

as $x \to \infty$. We now introduce an auxiliary function

$$\mathbf{D}(i\sqrt{x}) := \mathbf{A}(i\sqrt{x}) 2^2 \pi^{3/2} x^{u/2-3/4} \exp\{-\pi\sqrt{x} - i\pi(3/4-u/2) - 2i\sqrt{x}[\log(2/\sqrt{x})+1]\}. \tag{5.8}$$

The purpose of introducing this function is to cancel out the singularity in (5.5) by multiplying the exponential function inside the square brackets by $\mathbf{D}(-in\sqrt{t})$, that is, we write

$$W_n(n^2t) \sim (-1)^n \left(\frac{n}{e}\right)^{3n} 2^{\frac{1}{2}} t^{\frac{3}{4}-\frac{u}{2}} (1-t)^{-\frac{1}{4}} e^{n\pi\sqrt{t}}$$

$$\times \operatorname{Re}\left[\mathbf{D}(-in\sqrt{t}) \exp\left\{ i(2n+u-1) \arccos\sqrt{t}\right.\right.$$

$$\left.\left. + i2n\sqrt{t} \log \frac{\sqrt{t}}{1+\sqrt{1-t}} - i\frac{\pi}{4}\right\}\right]. \tag{5.9}$$

Note that, as $n \to \infty$, $\mathbf{D}(-in\sqrt{t}) \to 1$ uniformly for t bounded away from the origin. Thus, the above formula is asymptotically equivalent to (5.5) for any fixed $t \in (0,1)$. Moreover, if x is fixed and $t = x/n^2 = O(1/n^2)$, we can reduce the above formula to

$$W_n(x) \sim (-1)^n \left(\frac{n}{e}\right)^{3n} 2^{1/2} x^{3/4-u/2} n^{u-3/2} e^{\pi\sqrt{x}}$$

$$\times \operatorname{Re}\left[\mathbf{D}(-i\sqrt{x}) \exp\left\{ i(2n+u-1)(\pi/2 - \sqrt{x}/n)\right.\right.$$

$$\left.\left. + 2i\sqrt{x} \log \frac{\sqrt{x}/n}{2} - i\frac{\pi}{4}\right\}\right]. \tag{5.10}$$

Substituting the conjugate of (5.8) into the above formula gives (5.6). Thus, (5.9) is a uniform asymptotic formula which connects the two formulas (5.5) and (5.6). On account of (2.4) and (5.8), we can rewrite (5.9) as

$$W_n(n^2t) \sim \frac{2\mathbf{C}_n}{(1-t)^{1/4}} \operatorname{Re}\left[\mathbf{A}(-in\sqrt{t}) \exp\left\{ -i\left(n + \frac{u-1}{2}\right) \arccos(1-2t)\right.\right.$$

$$\left.\left. + 2in\sqrt{t} \log \frac{2e/n}{1+\sqrt{1-t}}\right\}\right]$$

$$\sim \frac{2\mathbf{C}_n |\mathbf{A}(in\sqrt{t})|}{(1-t)^{1/4}} \cos\left[\left(n + \frac{u-1}{2}\right) \arccos(1-2t)\right.$$

$$\left. + 2n\sqrt{t} \log \frac{1+\sqrt{1-t}}{2e/n} + \arg \mathbf{A}(in\sqrt{t})\right]. \tag{5.11}$$

Now, we let $x = n^2t = N^2T$. By Theorem 3, there exist two functions $C_1(x)$ and $C_2(x)$ such that

$$P_n(N^2T) = C_1(x)\mathcal{P}_n(N^2T) + C_2(x)\mathcal{Q}_n(N^2T),$$

where $\mathcal{P}_n$ and $\mathcal{Q}_n$ have the asymptotic expansions given in (4.1)–(4.2). On account of (1.4), (1.13), (2.2)–(2.3), and (2.4), we have $W_n(x) = \gamma_n K_n P_n(x)$, where

$$\gamma_n K_n \sim (-1)^n 2^{5/2} \pi^2 e^{-3n} n^{u-2+3n} = (-1)^n 2\mathbf{C}_n \sqrt{\pi}/n^{1/2}.$$

Thus, we have

$$W_n(N^2T) \sim \frac{2\mathbf{C}_n\sqrt{\pi}}{n^{1/2}t^{1/4}(1-t)^{1/4}}[C_1(x)\cos(N\eta(T)) + C_2(x)\sin(N\eta(T))]. \tag{5.12}$$

Recall that $N = n + \tau_0$ with $\tau_0 = (u-1)/2$. Since $x = N^2T = n^2t$ and

$$T = (n/N)^2 t = (1+\tau_0/n)^{-2}t \sim t - \frac{2\tau_0 t}{n},$$

we obtain from the definition of η in (4.3)

$$\begin{aligned} N\eta(T) &= (n+\tau_0)\arccos(1-2T) + n\sqrt{t}\log\frac{2-T+2\sqrt{1-T}}{T} \\ &= (n+\tau_0)\arccos(1-2t) - 2\tau_0\sqrt{\frac{t}{1-t}} \\ &\quad + n\sqrt{t}\log\frac{2-t+2\sqrt{1-t}}{t} + 2\tau_0\sqrt{\frac{t}{1-t}} + O(1/n) \\ &= \phi(n,t) - 2\sqrt{x}\log\frac{\sqrt{x}}{2e} + O(1/n), \end{aligned}$$

where

$$\phi(n,t) := (n+\tau_0)\arccos(1-2t) + 2n\sqrt{t}\log\frac{1+\sqrt{1-t}}{2e/n}. \tag{5.13}$$

We may rewrite (5.12) as

$$\begin{aligned} W_n(n^2t) \sim \frac{\mathbf{C}_n\sqrt{\pi}}{x^{1/4}(1-t)^{1/4}}&[(C_1(x)+iC_2(x))(2e/\sqrt{x})^{-2i\sqrt{x}}\exp\{-i\phi(n,t)\} \\ &+ (C_1(x)-iC_2(x))(2e/\sqrt{x})^{2i\sqrt{x}}\exp\{i\phi(n,t)\}]. \end{aligned}$$

Comparing the above formula with (5.11) gives

$$C_1(x) + iC_2(x) = \frac{x^{1/4}}{\sqrt{\pi}}(2e/\sqrt{x})^{2i\sqrt{x}}\mathbf{A}(-i\sqrt{x}),$$

$$C_1(x) - iC_2(x) = \frac{x^{1/4}}{\sqrt{\pi}}(2e/\sqrt{x})^{-2i\sqrt{x}}\mathbf{A}(i\sqrt{x}).$$

Solving the above equations yields

$$C_1(x) = \frac{x^{1/4}}{\sqrt{\pi}}\mathrm{Re}[(2e/\sqrt{x})^{2i\sqrt{x}}\mathbf{A}(-i\sqrt{x})], \tag{5.14}$$

$$C_2(x) = \frac{x^{1/4}}{\sqrt{\pi}}\mathrm{Im}[(2e/\sqrt{x})^{2i\sqrt{x}}\mathbf{A}(-i\sqrt{x})]. \tag{5.15}$$

From the asymptotic expansions of $\mathcal{P}_n$ and $\mathcal{Q}_n$ in Theorem 3, we have

$$W_n(N^2T) \sim \frac{(-1)^n \gamma_n K_n \sqrt{N}}{(1-T)^{1/4}\sqrt{\pi}} \Bigg\{ \mathrm{Re}[(2e/\sqrt{x})^{2i\sqrt{x}} \mathbf{A}(-i\sqrt{x})] \cos(N\eta(T)) \sum_{s=0}^{\infty} \frac{C_s(\eta)}{N^s} + \mathrm{Im}[(2e/\sqrt{x})^{2i\sqrt{x}} \mathbf{A}(-i\sqrt{x})] \sin(N\eta(T)) \sum_{s=0}^{\infty} \frac{D_s(\eta)}{N^s} \Bigg\};$$

note that in our case, $\alpha_0 = 4$. This proves (5.4).

5.2. *Airy-type approximation around the transition point* $t_2 = 1$

Following Wang and Wong [27], we set

$$\tau_0 = -\frac{\alpha_1 t_2 + \beta_1}{(2-\beta_0)\theta} = \frac{u-1}{2}, \quad N = n + \tau_0 = n + \frac{u-1}{2}, \tag{5.16}$$

and $\zeta(T)$ as given in (5.2)–(5.3); see (1.17) and [27, (4.10), p. 151]. For $0 < T < \infty$, two linearly independent solutions of (1.9) are given by

$$\mathcal{P}_n(N^2T) \sim N^{\frac{1}{6}} \left(\frac{\zeta}{4T(T-1)} \right)^{\frac{1}{4}} \times \left[\mathrm{Ai}(N^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} + N^{-\frac{1}{3}} \mathrm{Ai}'(N^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \right] \tag{5.17}$$

and

$$\mathcal{Q}_n(N^2T) \sim N^{\frac{1}{6}} \left(\frac{\zeta}{4T(T-1)} \right)^{\frac{1}{4}} \times \left[\mathrm{Bi}(N^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} + N^{-\frac{1}{3}} \mathrm{Bi}'(N^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \right], \tag{5.18}$$

where $A_0(\zeta) = 1$, $B_0(\zeta) = 0$, and all other coefficients could be determined in a recursive manner, see [27].

Remark 4. For simplicity of presentation, the authors of [27] have made an assumption that $-2 < \beta_0 < 2$. This assumption assures that one of the transition points is positive and the other is negative. Since in our case, we have a positive transition point, whether the other transition point is on the positive or negative real axis is irrelevant. For the case of a transition point occurring at the origin, we refer to [4]; see also Theorem 3 of the present paper.

Now, we have

$$P_n(x) \sim C_1(x)\mathcal{P}_n(x) + C_2(x)\mathcal{Q}_n(x), \tag{5.19}$$

where the coefficients $C_1(x)$ and $C_2(x)$ are functions of x and independent of n. These two coefficients can be determined by matching the asymptotic formulas in (5.17)–(5.19) with that in (3.1) on the interval $t > 1$. To this end, we recall the asymptotic approximations of the Airy functions and their derivatives:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{2}{3}z^{3/2}}, \quad \mathrm{Bi}(z) \sim \frac{1}{\sqrt{\pi} z^{1/4}} e^{\frac{2}{3}z^{3/2}} \tag{5.20}$$

as $z \to \infty$ and $|\arg z| \le \frac{\pi}{3} - \delta$; see [22, §9.7(ii)]. Note that from (5.2), we have

$$\frac{2}{3}[\zeta(T)]^{\frac{3}{2}} = \pi\sqrt{T} - \left[2\sqrt{T}\arcsin\frac{1}{\sqrt{T}} + \log(2T - 1 + 2\sqrt{T(T-1)})\right], \quad T \ge 1. \tag{5.21}$$

Hence, a combination of (5.17)–(5.21) gives

$$\begin{aligned} P_n(N^2T) &\sim \frac{C_1(x)}{2\sqrt{2\pi}}\left(\frac{1}{T(T-1)}\right)^{\frac{1}{4}} \exp\left\{N\left[2\sqrt{T}\arcsin\frac{1}{\sqrt{T}}\right.\right. \\ &\quad \left.\left. + \log(2T-1+2\sqrt{T(T-1)}) - \pi\sqrt{T}\right]\right\} + \frac{C_2(x)}{\sqrt{2\pi}}\left(\frac{1}{T(T-1)}\right)^{\frac{1}{4}} \\ &\quad \times \exp\left\{-N\left[2\sqrt{T}\arcsin\frac{1}{\sqrt{T}} + \log(2T-1+2\sqrt{T(T-1)}) - \pi\sqrt{T}\right]\right\}. \end{aligned} \tag{5.22}$$

In a similar manner, we have from (5.3) and [22, §9.7(ii)]

$$\frac{2}{3}[-\zeta(T)]^{\frac{3}{2}} = \pi - \arccos(1-2T) - \sqrt{T}\ln\frac{2-T+2\sqrt{1-T}}{T}, \quad 0 < T < 1, \tag{5.23}$$

and for $x > 0$,

$$\mathrm{Ai}(-x) \sim \frac{1}{\sqrt{\pi}x^{1/4}}\cos\left(\xi - \frac{\pi}{4}\right), \quad \mathrm{Bi}(-x) \sim -\frac{1}{\sqrt{\pi}x^{1/4}}\sin\left(\xi - \frac{\pi}{4}\right), \tag{5.24}$$

where $\xi = \frac{2}{3}x^{3/2}$. A combination of (5.17)–(5.19), (5.23), and (5.24) gives

$$\begin{aligned} P_n(N^2T) &\sim \frac{C_1(x)}{\sqrt{2\pi}}\left(\frac{1}{T(1-T)}\right)^{\frac{1}{4}} \cos\left\{N\left[\arccos(1-2T)\right.\right. \\ &\quad \left.\left. + \sqrt{T}\ln\frac{2-T+2\sqrt{1-T}}{T} - \pi\right] + \frac{\pi}{4}\right\} + \frac{C_2(x)}{\sqrt{2\pi}}\left(\frac{1}{T(1-T)}\right)^{\frac{1}{4}} \\ &\quad \times \sin\left\{N\left[\arccos(1-2T) + \sqrt{T}\ln\frac{2-T+2\sqrt{1-T}}{T} - \pi\right] + \frac{\pi}{4}\right\}. \end{aligned} \tag{5.25}$$

To find the value of the coefficient $C_1(x)$, we compare the expansion of $P_n(N^2T)$ in (5.22) with that of $\pi_n(n^2t)$ in (3.17). Note that both arguments in P_n and π_n are equal to x. But, since $n \neq N$, the variables t and T cannot be the same. To make this distinction, we use the asymptotic relationship

$$T = \left(\frac{n}{N}\right)^2 t = \left(\frac{n}{n + \dfrac{u-1}{2}}\right)^2 t = \left(1 + \frac{u-1}{2n}\right)^{-2} t \sim \left(1 - \frac{u-1}{n}\right) t \tag{5.26}$$

to recast the result in (5.22) in terms of t:

$$\begin{aligned} P_n(N^2T) \sim \frac{C_1(x)}{2\sqrt{2\pi}} \left(\frac{1}{t(t-1)}\right)^{\frac{1}{4}} \exp\Bigg\{ & n2\sqrt{t} \arcsin \frac{1}{\sqrt{t}} + (u-1)\sqrt{\frac{t}{t-1}} \\ & + \left(n + \frac{u-1}{2}\right) \log(2t - 1 + 2\sqrt{t(t-1)}) - (u-1)\sqrt{\frac{t}{t-1}} - \pi n \sqrt{t} \Bigg\} \\ & + \frac{C_2(x)}{\sqrt{2\pi}} \cdots . \end{aligned} \tag{5.27}$$

In view of (1.13), (2.19), and (5.27), we have

$$\frac{\left(\dfrac{n}{2e}\right)^{2n} (4t)^{-\frac{u-1}{2}} \left(\dfrac{t}{t-1}\right)^{\frac{1}{4}}}{(4\pi)^2 2^{-u} n^{u-2} \left(\dfrac{n}{2e}\right)^{2n}} = \frac{C_1(x)}{2\sqrt{2\pi}} \left(\frac{1}{t(t-1)}\right)^{\frac{1}{4}} \exp(-\pi n \sqrt{t}), \tag{5.28}$$

which in turn gives

$$C_1(x) = (2\pi)^{-3/2} x^{1-\frac{u}{2}} \exp(\pi\sqrt{x}). \tag{5.29}$$

To find $C_2(x)$, we compare (5.25) with (3.20) and readily obtain $C_2(x) = 0$. From (5.19), it follows that

$$\begin{aligned} P_n(N^2T) \sim \frac{1}{(2\pi)^{3/2}} x^{1-\frac{u}{2}} e^{\pi\sqrt{x}} N^{\frac{1}{6}} \left(\frac{\zeta}{4T(T-1)}\right)^{\frac{1}{4}} & \\ \times \left[\mathrm{Ai}(N^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} + N^{-\frac{1}{3}} \mathrm{Ai}'(N^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \right] & . \end{aligned} \tag{5.30}$$

Recall that the Wilson polynomials are orthogonal with respect to the weight function $w(x)$ given in (1.2). From Stirling's formula [22, (5.11.9)], we have

$$[w(x)]^{\frac{1}{2}} \sim (2\pi)^{\frac{3}{2}} x^{\frac{u}{2}-1} e^{-\pi\sqrt{x}}. \tag{5.31}$$

In view of (1.4) and (1.13), this proves (5.1) and completes the proof of Theorem 4.

6. Plancherel-Type Asymptotics and Asympotics of Zeros for $W_n(x)$

Corollary 1 (Plancherel–Rotach-type asymptotics). *Let $N = n + \frac{1}{2}(u-1)$ and $T = 1 + N^{-2/3}s$. Then, as $n \to \infty$,*

$$[w(N^2T)]^{\frac{1}{2}} W_n(N^2T) = (-1)^n 2^{5/2}\pi^2 e^{-3n} n^{u-2+3n} \frac{1}{\sqrt{2}} N^{\frac{1}{6}} [\mathrm{Ai}(s) + O(N^{-2/3})] \tag{6.1}$$

for any fixed s.

Proof. We have the following asymptotic formula:

$$\zeta(T) = (T-1) - \frac{1}{3}(T-1)^2 + \mathcal{O}\left((T-1)^3\right) \quad \text{as } T \to 1. \tag{6.2}$$

Let $T = 1 + N^{-2/3}s$ in (5.1), we have

$$[w(N^2T)]^{\frac{1}{2}} P_n(N^2T) = \frac{1}{\sqrt{2}} N^{\frac{1}{6}} [\mathrm{Ai}(s) + O(N^{-2/3})] \quad \text{as } N \to \infty$$

for any fixed s. □

By using a similar argument as in [7], we have the following result.

Corollary 2 (Asymptotics for zeros). *Let $x_{n,k}$ be the zeros of $W_n(x)$ such that $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$, and let $\mathfrak{a}_k$ be the zeros of the Airy function $\mathrm{Ai}(x)$ in the descending order. Then, for fixed k and large n, we have*

$$x_{n,n-k+1} = N^2 + \mathfrak{a}_k N^{\frac{4}{3}} + O(N^{2/3}), \tag{6.3}$$

where $N = n + \frac{1}{2}(u-1)$.

Remark 5. To find the asymptotic behaviors of the smallest zeros $x_{n,k}$ with fixed k and large n, we shall make use of fixed-x asymptotics in (2.6) with $x > 0$, or equivalently, (2.16). Let $\mathfrak{b}_k$ be the positive zeros of $\mathrm{Re}[n^{2i\sqrt{x}}\mathbf{A}(i\sqrt{x})]$ in the ascending order. Then, by using a similar argument as in [7], we have $x_{n,k} = \mathfrak{b}_k + O(n^{-1})$ for fixed k and large n.

Acknowledgments

We are very grateful to the anonymous referee for his/her careful reading and valuable suggestions which have helped to improve the presentation of this paper. This work was supported in part by the National Natural Science Foundation of China under Grant No. 11801480 and by the President's Fund from Chinese University of Hong Kong, Shenzhen.

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