Communications in **M**athematical **A**nalysis
Volume 7, Number 1, pp. 50–54 (2009)
ISSN 1938-9787
www.commun-math-anal.org

ASYMPTOTICS OF THE q-THETA FUNCTION

X. S. WANG*
Department of Mathematics
University of Science and Technology of China
Hefei, Anhui 230026, P. R. China
Department of Mathematics
City University of Hong Kong
Tat Chee Avenue, Kowloon, Hong Kong SAR
Joint Advanced Research Center
University of Science and Technology of China-City University of Hong Kong
Suzhou, Jiangshu 215123, P. R. China

(Communicated by Toka Diagana)

Abstract

An asymptotic formula is given to the q-Theta function

$$\Theta_q(x) := \sum_{k=-\infty}^{\infty} q^{k^2} x^k$$

as $q \to 1^-$, where $x > 0$ is fixed.

AMS Subject Classification: 41A60; 33D99.

Keywords: asymptotics; q-Theta function; Jacobi Theta functions; asymptotic formula; asymptotically equal.

1 Introduction

For $0 < q < 1$ and $x \in \mathbb{C}$, the q-Theta function is defined by [2]

$$\Theta_q(x) := \sum_{k=-\infty}^{\infty} q^{k^2} x^k. \tag{1.1}$$

It satisfies the Jacobi triple product identity [1, Theorem 12.3.2]

$$\Theta_q(x) = \prod_{k=0}^{\infty} (1-q^{2k+2})(1+q^{2k+1}x)(1+q^{2k+1}/x). \tag{1.2}$$

*E-mail address: xswang4@mail.ustc.edu.cn

The q-Theta function can be written in terms of the Jacobi Theta functions [5, Chapter 21]. For example, by choosing any $z \in \mathbb{C}$ such that $x = e^{2iz}$, we have

$$\Theta_q(x) = \vartheta_3(z, q). \tag{1.3}$$

For more properties of the q-Theta function and the Jacobi Theta functions, please refer to [5, Chapter 21] and references therein.

Note that the definition (1.1) and the Jacobi triple product (1.2) of the q-Theta function are also valid for $q \in \mathbb{C}$ with $|q| < 1$. However, as in the theory of q-orthogonal polynomials (or basic hypergeometric orthogonal polynomials), we will always assume $0 < q < 1$; see [1] and [3]. With the aid of the q-Theta function, Ismail and Zhang [2] derive several asymptotic formulas for three classes of q-orthogonal polynomials. Their results have been improved by Wang and Wong in [4], where again, the q-Theta function plays a significant role. Therefore, it will be useful to investigate asymptotic behavior of the q-Theta function in a stand-alone manner. This paper is dedicated to give an asymptotic formula for the q-Theta function as $q \to 1^-$ with fixed $x > 0$. As far as we are aware, this result has not been obtained previously.

2 Main Results

Our main theorem is stated below.

Theorem 2.1. *As $q \to 1^-$, we have*

$$\Theta_q(x) \sim \sqrt{\frac{\pi}{-\ln q}} \exp\{\frac{(\ln x)^2}{-4\ln q}\} \tag{2.1}$$

for $x > 0$. Here the symbol "$\sim$" means asymptotically equal, that is, we write $A_q \sim B_q$ if $\lim\limits_{q\to 1^-} A_q/B_q = 1$.

For preparation, we study the sum

$$I(\lambda, a) := \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \tag{2.2}$$

for $a \in \mathbb{R}$ and $\lambda > 0$. It is easily seen from (1.1) and (2.2) that

$$\Theta_q(x) = I(-1/\ln q, \ln x) + I(-1/\ln q, -\ln x) - 1. \tag{2.3}$$

Lemma 2.2. *As $\lambda \to +\infty$,*

$$I(\lambda, 0) := \sum_{k=0}^{\infty} e^{-k^2/\lambda} \sim \sqrt{\pi\lambda}/2. \tag{2.4}$$

If the integer-valued function $N = N(\lambda) \in \mathbb{N}$ satisfies

$$\lim_{\lambda\to+\infty} N/\lambda = c$$

for some positive constant $c > 0$, *we have*

$$\sum_{k=0}^{N} e^{-k^2/\lambda} \sim \sqrt{\pi\lambda}/2 \tag{2.5}$$

as $\lambda \to +\infty$.

Proof. Consider the auxiliary integral

$$\widetilde{I}(\lambda) := \int_0^\infty e^{-t^2/\lambda} dt = \sum_{k=0}^{\infty} \int_k^{k+1} e^{-t^2/\lambda} dt.$$

Since $e^{-(k+1)^2/\lambda} \leq e^{-t^2/\lambda} \leq e^{-k^2/\lambda}$ for $k \leq t \leq k+1$, it follows that

$$\sum_{k=1}^{\infty} e^{-k^2/\lambda} \leq \widetilde{I}(\lambda) \leq \sum_{k=0}^{\infty} e^{-k^2/\lambda}.$$

On account of (2.2), we have $\widetilde{I}(\lambda) \leq I(\lambda, 0) \leq \widetilde{I}(\lambda) + 1$. Multiply this by $\lambda^{-1/2}$ and then let $\lambda \to +\infty$. Formula (2.4) follows from the fact $\widetilde{I}(\lambda) = \sqrt{\pi\lambda}/2$.

To prove (2.5), we shall estimate the sum

$$\sum_{k=N+1}^{\infty} e^{-k^2/\lambda} = \sum_{k=1}^{\infty} e^{-(k+N)^2/\lambda} \leq \sum_{k=0}^{\infty} e^{-N^2/\lambda - 2Nk/\lambda} = \frac{e^{-N^2/\lambda}}{1 - e^{-2N/\lambda}}.$$

As $\lambda \to +\infty$, the right-hand side of the last inequality vanishes since $N/\lambda \to c > 0$ by assumption. This implies

$$\lim_{\lambda \to +\infty} \sum_{k=N+1}^{\infty} e^{-k^2/\lambda} = 0.$$

Therefore, formula (2.5) follows from (2.4). □

Lemma 2.3. *For* $a < 0$, *we have*

$$I(\lambda, a) := \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \sim (1 - e^a)^{-1} \tag{2.6}$$

as $\lambda \to +\infty$.

Proof. Since $k^2/\lambda \leq 1/\sqrt{\lambda}$ for $0 \leq k \leq \lfloor \lambda^{1/4} \rfloor$, we have

$$I(\lambda, a) = \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \geq \sum_{k=0}^{\lfloor \lambda^{1/4} \rfloor} e^{-1/\sqrt{\lambda} + ak} = \frac{e^{-1/\sqrt{\lambda}}(1 - e^{a(\lfloor \lambda^{1/4} \rfloor + 1)})}{1 - e^a}.$$

By letting $\lambda \to +\infty$, we obtain from the assumption $a < 0$ that

$$\liminf_{\lambda \to +\infty} I(\lambda, a) \geq (1 - e^a)^{-1}.$$

Moreover, it is easily seen that

$$I(\lambda, a) = \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \leq \sum_{k=0}^{\infty} e^{ak} = (1 - e^a)^{-1}.$$

Coupling the last two inequalities yields our desired result. □

Lemma 2.4. *For $a > 0$, we have*

$$I(\lambda, a) := \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \sim \sqrt{\pi\lambda} e^{\lambda a^2/4} \tag{2.7}$$

as $\lambda \to +\infty$.

Proof. Consider the sum

$$e^{-\lambda a^2/4} I(\lambda, a) = \sum_{k=0}^{\infty} e^{-(k-\lambda a/2)^2/\lambda} = \sum_{k=0}^{m} + \sum_{k=m+1}^{\infty} =: I_1 + I_2, \tag{2.8}$$

where $m := \lfloor \lambda a/2 \rfloor$. We intend to show $I_1 \sim \sqrt{\pi\lambda}/2$ and $I_2 \sim \sqrt{\pi\lambda}/2$ as $\lambda \to +\infty$.

Firstly, it follows from $m \le \lambda a/2 \le m+1$ that

$$I_1 := \sum_{k=0}^{m} e^{-(k-\lambda a/2)^2/\lambda} \le \sum_{k=0}^{m} e^{-(m-k)^2/\lambda} = \sum_{k=0}^{m} e^{-k^2/\lambda},$$

and

$$I_1 \ge \sum_{k=0}^{m} e^{-(m+1-k)^2/\lambda} = \sum_{k=0}^{m+1} e^{-k^2/\lambda} - 1.$$

Since $m/\lambda \to a/2 > 0$ as $\lambda \to +\infty$, we obtain from (2.5) and the last two inequalities that

$$\lim_{\lambda \to +\infty} I_1/\sqrt{\lambda} = \sqrt{\pi}/2. \tag{2.9}$$

Secondly, since $m \le \lambda a/2 \le m+1$, we have

$$I_2 := \sum_{k=m+1}^{\infty} e^{-(k-\lambda a/2)^2/\lambda} \le \sum_{k=m+1}^{\infty} e^{-(k-m-1)^2/\lambda} = \sum_{k=0}^{\infty} e^{-k^2/\lambda},$$

and

$$I_2 \ge \sum_{k=m+1}^{\infty} e^{-(k-m)^2/\lambda} = \sum_{k=0}^{\infty} e^{-k^2/\lambda} - 1.$$

applying (2.4) to the last two inequalities gives

$$\lim_{\lambda \to +\infty} I_2/\sqrt{\lambda} = \sqrt{\pi}/2. \tag{2.10}$$

Finally, a combination of (2.8)-(2.10) yields (2.7) immediately. □

Proof of Theorem 2.1. For $x = 1$, we obtain from (2.3) that

$$\Theta_q(1) = 2I(-1/\ln q, 0) - 1.$$

Coupling this and (2.4) gives

$$\lim_{q \to 1^-} \Theta_q(1) \sqrt{-\ln q} = \sqrt{\pi}. \tag{2.11}$$

For $x > 1$, it follows from (2.6) that

$$\lim_{q\to 1^-}[I(-1/\ln q, -\ln x) - 1]\sqrt{-\ln q}\exp\{\frac{(\ln x)^2}{4\ln q}\} = 0.$$

On the other hand, from (2.7) we have

$$\lim_{q\to 1^-} I(-1/\ln q, \ln x)\sqrt{-\ln q}\exp\{\frac{(\ln x)^2}{4\ln q}\} = \sqrt{\pi}.$$

Therefore, applying the last two equations to (2.3) yields

$$\lim_{q\to 1^-} \Theta_q(x)\sqrt{-\ln q}\exp\{\frac{(\ln x)^2}{4\ln q}\} = \sqrt{\pi}. \tag{2.12}$$

Similarly, for $0 < x < 1$, a combination of (2.3), (2.6) and (2.7) implies

$$\lim_{q\to 1^-} \Theta_q(x)\sqrt{-\ln q}\exp\{\frac{(\ln x)^2}{4\ln q}\} = \sqrt{\pi}. \tag{2.13}$$

Thus, formula (2.1) follows from (2.11)-(2.13). This ends the proof of Theorem 2.1. □

Acknowledgments

The author thanks the referees for their careful reading of the manuscript and helpful comments.

References

[1] M. E. H. Ismail, *"Classical and Quantum Orthogonal Polynomials in One Variable"*, Cambridge University Press, Cambridge, 2005.

[2] M. E. H. Ismail and R. M. Zhang, Chaotic and periodic asymptotics for q-orthogonal polynomials, *Int. Math. Res. Not.* 2006, Art. ID 83274, 33 pp.

[3] R. Koekoek, R. F. Swarttouw, *"The Askey-scheme of Hypergeometric Orthogonal Polynomials and its q-analogue"*, Report no. 98-17, TU-Delft, 1998.

[4] X. S. Wang and R. Wong, Discrete analogues of Laplace's approximation, *Asymptotic Analysis* **54** (2007), pp 165-180.

[5] E. T. Whittaker and G. N. Watson, *"A Course of Modern Analysis"*, fourth edition, Cambridge University Press, Cambridge, 1927.