ASYMPTOTICS OF THE $q$-THETA FUNCTION

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Abstract

An asymptotic formula is given to the $q$-Theta function

$$\Theta_q(x) := \sum_{k=-\infty}^{\infty} q^{k^2} x^k$$

as $q \to 1^-$, where $x > 0$ is fixed.

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1 Introduction

For $0 < q < 1$ and $x \in \mathbb{C}$, the $q$-Theta function is defined by [2]

$$\Theta_q(x) := \sum_{k=-\infty}^{\infty} q^{k^2} x^k. \quad (1.1)$$

It satisfies the Jacobi triple product identity [1, Theorem 12.3.2]

$$\Theta_q(x) = \prod_{k=0}^{\infty} (1 - q^{2k+2})(1 + q^{2k+1})^{1/(1+q^{2k+1}/x)}. \quad (1.2)$$

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The \( q \)-Theta function can be written in terms of the Jacobi Theta functions [5, Chapter 21]. For example, by choosing any \( z \in \mathbb{C} \) such that \( x = e^{2iz} \), we have

\[
\Theta_q(x) = \vartheta_3(z, q).
\]

(1.3)

For more properties of the \( q \)-Theta function and the Jacobi Theta functions, please refer to [5, Chapter 21] and references therein.

Note that the definition (1.1) and the Jacobi triple product (1.2) of the \( q \)-Theta function are also valid for \( q \in \mathbb{C} \) with \( |q| < 1 \). However, as in the theory of \( q \)-orthogonal polynomials (or basic hypergeometric orthogonal polynomials), we will always assume \( 0 < q < 1 \); see [1] and [3]. With the aid of the \( q \)-Theta function, Ismail and Zhang [2] derive several asymptotic formulas for three classes of \( q \)-orthogonal polynomials. Their results have been improved by Wang and Wong in [4], where again, the \( q \)-Theta function plays a significant role. Therefore, it will be useful to investigate asymptotic behavior of the \( q \)-Theta function in a stand-alone manner. This paper is dedicated to give an asymptotic formula for the \( q \)-Theta function as \( q \to 1^- \) with fixed \( x > 0 \). As far as we are aware, this result has not been obtained previously.

2 Main Results

Our main theorem is stated below.

**Theorem 2.1.** As \( q \to 1^- \), we have

\[
\Theta_q(x) \sim \sqrt{\frac{\pi}{-\ln q}} \exp\left\{ \frac{(\ln x)^2}{-4 \ln q} \right\}
\]

(2.1)

for \( x > 0 \). Here the symbol “\( \sim \)” means asymptotically equal, that is, we write \( A_q \sim B_q \) if

\[
\lim_{q \to 1^-} A_q / B_q = 1.
\]

For preparation, we study the sum

\[
I(\lambda, a) := \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak}
\]

(2.2)

for \( a \in \mathbb{R} \) and \( \lambda > 0 \). It is easily seen from (1.1) and (2.2) that

\[
\Theta_q(x) = I(-1/\ln q, \ln x) + I(-1/\ln q, -\ln x) - 1.
\]

(2.3)

**Lemma 2.2.** As \( \lambda \to +\infty \),

\[
I(\lambda, 0) := \sum_{k=0}^{\infty} e^{-k^2/\lambda} \sim \sqrt{\pi \lambda/2}.
\]

(2.4)

If the integer-valued function \( N = N(\lambda) \in \mathbb{N} \) satisfies

\[
\lim_{\lambda \to +\infty} N/\lambda = c
\]
for some positive constant \( c > 0 \), we have

\[
\sum_{k=0}^{N} e^{-k^2/\lambda} \sim \sqrt{\pi \lambda}/2
\]  

(2.5)
as \( \lambda \to +\infty \).

**Proof.** Consider the auxiliary integral

\[
\bar{I}(\lambda) := \int_{0}^{\infty} e^{-t^2/\lambda} dt = \sum_{k=0}^{\infty} \int_{k}^{k+1} e^{-t^2/\lambda} dt.
\]

Since \( e^{-(k+1)^2/\lambda} \leq e^{-t^2/\lambda} \leq e^{-k^2/\lambda} \) for \( k \leq t \leq k+1 \), it follows that

\[
\sum_{k=1}^{\infty} e^{-k^2/\lambda} \leq \bar{I}(\lambda) \leq \sum_{k=0}^{\infty} e^{-k^2/\lambda}.
\]

On account of (2.2), we have \( \bar{I}(\lambda) \leq I(\lambda, 0) \leq \bar{I}(\lambda) + 1 \). Multiply this by \( \lambda^{-1/2} \) and then let \( \lambda \to +\infty \). Formula (2.4) follows from the fact \( \bar{I}(\lambda) = \sqrt{\pi \lambda}/2 \).

To prove (2.5), we shall estimate the sum

\[
\sum_{k=N+1}^{\infty} e^{-k^2/\lambda} = \sum_{k=1}^{\infty} e^{-(k+N)^2/\lambda} \leq \sum_{k=0}^{\infty} e^{-N^2/\lambda - 2Nk/\lambda} = \frac{e^{-N^2/\lambda}}{1 - e^{-2N/\lambda}}.
\]

As \( \lambda \to +\infty \), the right-hand side of the last inequality vanishes since \( N/\lambda \to c > 0 \) by assumption. This implies

\[
\lim_{\lambda \to +\infty} \sum_{k=N+1}^{\infty} e^{-k^2/\lambda} = 0.
\]

Therefore, formula (2.5) follows from (2.4). \( \square \)

**Lemma 2.3.** For \( a < 0 \), we have

\[
I(\lambda, a) := \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \sim (1 - e^a)^{-1}
\]  

(2.6)
as \( \lambda \to +\infty \).

**Proof.** Since \( k^2/\lambda \leq 1/\sqrt{\lambda} \) for \( 0 \leq k \leq [\lambda^{1/4}] \), we have

\[
I(\lambda, a) = \sum_{k=0}^{[\lambda^{1/4}]} e^{-k^2/\lambda + ak} \geq \sum_{k=0}^{\lfloor \lambda^{1/4} \rfloor} e^{-1/\sqrt{\lambda} + ak} = \frac{e^{-1/\sqrt{\lambda}}(1 - e^{a([\lambda^{1/4}]+1)})}{1 - e^a}.
\]

By letting \( \lambda \to +\infty \), we obtain from the assumption \( a < 0 \) that

\[
\liminf_{\lambda \to +\infty} I(\lambda, a) \geq (1 - e^a)^{-1}.
\]

Moreover, it is easily seen that

\[
I(\lambda, a) = \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \leq \sum_{k=0}^{\infty} e^{ak} = (1 - e^a)^{-1}.
\]

Coupling the last two inequalities yields our desired result. \( \square \)
Lemma 2.4. For $a > 0$, we have

$$I(\lambda, a) := \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \sim \sqrt{\pi\lambda} e^{\lambda a^2/4}$$  \hspace{1cm} (2.7)

as $\lambda \to +\infty$.

Proof. Consider the sum

$$e^{-\lambda a^2/4}I(\lambda, a) = \sum_{k=0}^{\infty} e^{-(k-\lambda a/2)^2/\lambda} = \sum_{k=0}^{m} + \sum_{k=m+1}^{\infty} =: I_1 + I_2,$$  \hspace{1cm} (2.8)

where $m := \lfloor \lambda a/2 \rfloor$. We intend to show $I_1 \sim \sqrt{\pi\lambda}/2$ and $I_2 \sim \sqrt{\pi\lambda}/2$ as $\lambda \to +\infty$.

Firstly, it follows from $m \leq \lambda a/2 \leq m + 1$ that

$$I_1 := \sum_{k=0}^{m} e^{-(k-\lambda a/2)^2/\lambda} \leq \sum_{k=0}^{m} e^{-(m-k)^2/\lambda} = \sum_{k=0}^{m} e^{-k^2/\lambda},$$

and

$$I_1 \geq \sum_{k=0}^{m} e^{-(m+1-k)^2/\lambda} = \sum_{k=0}^{m+1} e^{-k^2/\lambda} - 1.$$  

Since $m/\lambda \to a/2 > 0$ as $\lambda \to +\infty$, we obtain from (2.5) and the last two inequalities that

$$\lim_{\lambda \to +\infty} I_1/\sqrt{\lambda} = \sqrt{\pi}/2. \hspace{1cm} (2.9)$$

Secondly, since $m \leq \lambda a/2 \leq m + 1$, we have

$$I_2 := \sum_{k=m+1}^{\infty} e^{-(k-\lambda a/2)^2/\lambda} \leq \sum_{k=m+1}^{\infty} e^{-(k-m-1)^2/\lambda} = \sum_{k=0}^{\infty} e^{-k^2/\lambda},$$

and

$$I_2 \geq \sum_{k=m+1}^{\infty} e^{-(k-m)^2/\lambda} = \sum_{k=0}^{\infty} e^{-k^2/\lambda} - 1.$$  

applying (2.4) to the last two inequalities gives

$$\lim_{\lambda \to +\infty} I_2/\sqrt{\lambda} = \sqrt{\pi}/2. \hspace{1cm} (2.10)$$

Finally, a combination of (2.8)-(2.10) yields (2.7) immediately. \hfill \Box

Proof of Theorem 2.1. For $x = 1$, we obtain from (2.3) that

$$\Theta_q(1) = 2I(-1/\ln q, 0) - 1.$$  

Coupling this and (2.4) gives

$$\lim_{q \to 1^-} \Theta_q(1)\sqrt{-\ln q} = \sqrt{\pi}. \hspace{1cm} (2.11)$$
For $x > 1$, it follows from (2.6) that
\[
\lim_{q \to 1^-} \left[ I(-1/\ln q, -\ln x) - 1 \right] \sqrt{-\ln q} \exp \left\{ \frac{(\ln x)^2}{4 \ln q} \right\} = 0.
\]
On the other hand, from (2.7) we have
\[
\lim_{q \to 1^-} I(-1/\ln q, \ln x) \sqrt{-\ln q} \exp \left\{ \frac{(\ln x)^2}{4 \ln q} \right\} = \sqrt{\pi}.
\]
Therefore, applying the last two equations to (2.3) yields
\[
\lim_{q \to 1^-} \Theta_q(x) \sqrt{-\ln q} \exp \left\{ \frac{(\ln x)^2}{4 \ln q} \right\} = \sqrt{\pi}.
\tag{2.12}
\]
Similarly, for $0 < x < 1$, a combination of (2.3), (2.6) and (2.7) implies
\[
\lim_{q \to 1^-} \Theta_q(x) \sqrt{-\ln q} \exp \left\{ \frac{(\ln x)^2}{4 \ln q} \right\} = \sqrt{\pi}.
\tag{2.13}
\]
Thus, formula (2.1) follows from (2.11)-(2.13). This ends the proof of Theorem 2.1. □

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References


