

Global stability analysis of a delayed susceptible–infected–susceptible epidemic model

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We study a susceptible–infected–susceptible model with distributed delays. By constructing suitable Lyapunov functionals, we demonstrate that the global dynamics of this model is fully determined by the basic reproductive ratio R_0 . To be specific, we prove that if $R_0 \leq 1$, then the disease-free equilibrium is globally asymptotically stable. On the other hand, if $R_0 > 1$, then the endemic equilibrium is globally asymptotically stable. It is remarkable that the model dynamics is independent of the probability of immunity lost.

Keywords: SIS model; distributed delay; global stability; Lyapunov functional; Lyapunov–LaSalle invariance principle

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1. Introduction

Let $S(t)$ and $I(t)$ be the density of susceptible and infected population at time t , respectively. We consider the following delayed SIS (susceptible–infected–susceptible) model:

$$S'(t) = b - \beta S(t)I(t) - \mu_S S(t) + q\gamma I(t), \quad (1)$$

$$I'(t) = \int_0^\infty p(\tau) e^{-\mu\tau} \beta S(t-\tau)I(t-\tau) d\tau - \mu_I I(t) - \gamma I(t), \quad (2)$$

where $b > 0$ denotes a constant birth rate, $\beta > 0$ is the disease transmission rate, $\mu_S > 0$ and $\mu_I > 0$ stand for the death rates of susceptible and infected individuals, respectively. $p(\tau) \geq 0$ with $\tau \in [0, \infty)$ is the probability density function of transmission delay, $\mu \geq 0$ corresponds to the death rate during latent period, $\gamma \geq 0$ is the recovery rate of infected individuals, and $q \in [0, 1]$ denotes the probability of immunity lost.

We remark that the delayed nonlinear incidence rates adopted in our model are different from those in [3,4,6,9]. In those models, the incidence rates take the same form $\beta S(t)I(t-\tau)$ in both equations of $S'(t)$ and $I'(t)$. In our model, the incidence rates take the form $\beta S(t)I(t)$ in the equation of $S'(t)$, and a delayed form $\beta S(t-\tau)I(t-\tau)$ in the equation of $I'(t)$. It seems reasonable to

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assume that the susceptible individuals at time t can only contact the infected individuals at the same time t , not the past time $t - \tau$. On the other hand, the infected individuals at the past time $t - \tau$ can only contact susceptible individuals at $t - \tau$, instead of t . In our model, we assume that the susceptible individuals are immediately removed from the susceptible group if they are infected, and they will enter the infected group after a certain latent period. Therefore, the delayed form of nonlinear incidence rate $\beta S(t - \tau)I(t - \tau)$ only appears in the equation of $I'(t)$.

The SIS model (1)–(2) always admits a disease-free equilibrium $(S_0, 0)$ with $S_0 := b/\mu_S$. Define the basic reproductive ratio

$$R_0 := \frac{b\beta_1}{\mu_S(\mu_I + \gamma)}, \quad (3)$$

where

$$\beta_1 := \beta \int_0^\infty p(\tau) e^{-\mu\tau} d\tau \leq \beta. \quad (4)$$

If $R_0 > 1$, the model possesses a unique endemic equilibrium (S^*, I^*) , where

$$S^* := \frac{\mu_I + \gamma}{\beta_1}, \quad (5)$$

$$I^* := \frac{b - \mu_S S^*}{\beta S^* - q\gamma} = \frac{b\beta_1 - \mu_S(\mu_I + \gamma)}{\beta(\mu_I + \gamma) - q\beta_1\gamma} = \frac{\mu_S(\mu_I + \gamma)(R_0 - 1)}{\beta(\mu_I + \gamma) - q\beta_1\gamma}. \quad (6)$$

Since $\beta(\mu_I + \gamma) - q\beta_1\gamma > \beta\gamma - q\beta_1\gamma \geq 0$, it is readily seen that $I^* > 0$ if and only if $R_0 > 1$.

Our objective is to prove that R_0 is the threshold parameter for the global dynamics of the SIS model (1)–(2). We shall construct two suitable Lyapunov functionals which are modified from those in [3,4,6,7,9]. Our result is more decent in the sense that no additional condition is required to obtain global stability of endemic equilibrium; while in the literature (cf. [3,4]), a technical assumption that the probability of immunity lost should be small is needed. Our main results are given in Section 2. We will conclude this paper with a discussion in Section 3.

2. Results

Throughout this paper, we assume that the probability density function $p(\tau)$ satisfies

$$\int_0^\infty p(\tau) e^{\lambda\tau} d\tau < \infty$$

for some $\lambda > 0$. The suitable state space for our system (1)–(2) is the Banach space X (see [1] for example) which consists of all continuous functions $(x^1, x^2) \in C((-\infty, 0], \mathbb{R}^2)$ such that $x^1(\theta) e^{\lambda\theta}$ and $x^2(\theta) e^{\lambda\theta}$ are uniformly continuous for $\theta \in (-\infty, 0]$, and that

$$\|(x^1, x^2)\|_X := \sup_{\theta \leq 0} (|x^1(\theta)| + |x^2(\theta)|) e^{\lambda\theta} < \infty.$$

Here, $\|\cdot\|_X$ denotes the weighted norm of X . For a function $\phi \in C((-\infty, t], \mathbb{R})$, we denote $\phi_t \in C((-\infty, 0], \mathbb{R})$ such that $\phi_t(\theta) := \phi(t + \theta)$ for $\theta \in (-\infty, 0]$. It follows from the standard theory of well-posedness for functional differential equations [5] that given any initial conditions $x_0 = (x_0^1, x_0^2) \in X$, the system (1)–(2) has a unique solution $x_t = (x_t^1, x_t^2) \in X$ for any $t > 0$.

We now show that the solutions of (1)–(2) are non-negative if the initial values are non-negative.

PROPOSITION 2.1 *Given the initial values such that $S(t) \geq 0$ and $I(t) \geq 0$ for all $t \leq 0$, we have $S(t) > 0$ and $I(t) \geq 0$ for all $t > 0$. If, in addition, $S(t)I(t) > 0$ for all $t \leq 0$, then $I(t) > 0$ for all $t \geq 0$.*

Proof First, we claim that $S(t)$ and $I(t)$ are non-negative for all $t > 0$. If, in contrary, there exists a $t_0 \geq 0$ such that $(S(t), I(t))$ leaves the first quadrant at the first time, we have either (i) $S(t_0) = 0$ and $S'(t_0) < 0$; or (ii) $I(t_0) = 0$ and $I'(t_0) < 0$. Moreover, $S(t) \geq 0$ and $I(t) \geq 0$ for all $t \leq t_0$. However, case (i) contradicts Equation (1); while case (ii) contradicts Equation (2).

Next, we show that $S(t)$ is strictly positive for all $t > 0$. Assume $S(t_1) = 0$ for some $t_1 > 0$. Since $S(t) \geq 0$ for all t , it follows that $t = t_1$ is a critical point of $S(t)$ and thus $S'(t_1) = 0$. On the other hand, we obtain from (1) that $S'(t_1) = b + q\gamma I(t_1) \geq b > 0$, a contradiction.

Finally, if, in addition, $S(t)I(t) > 0$ for all $t \leq 0$, we prove by contradiction that $I(t) > 0$ for all $t > 0$. Assume t_2 is the first time when $I(t)$ losses its positiveness, we have $I(t_2) = I'(t_2) = 0$ and $I(t) > 0$ for all $t < t_2$, which again contradict Equation (2). ■

Our main theorem is given as below.

THEOREM 2.2 *If $R_0 \leq 1$, then the disease-free equilibrium $(S_0, 0)$ of (1)–(2) is globally asymptotically stable; if $R_0 > 1$, then the endemic equilibrium (S^*, I^*) of (1)–(2) is globally asymptotically stable.*

Proof If $R_0 \leq 1$, we construct the Lyapunov functional $U : X \rightarrow \mathbb{R}$ as

$$U(x^1, x^2) := \frac{\beta_1 S_0}{\beta S_0 - q\gamma} [x^1(0) - S_0 \ln x^1(0)] + x^2(0) + \int_0^\infty \int_{-\tau}^0 p(\tau) e^{-\mu\tau} \beta x^1(\theta) x^2(\theta) d\theta d\tau.$$

Restricting U along a solution (S, I) of the system (1)–(2), we have

$$U(t) = \frac{\beta_1 S_0}{\beta S_0 - q\gamma} [S(t) - S_0 \ln S(t)] + I(t) + \int_0^\infty \int_{t-\tau}^t p(\tau) e^{-\mu\tau} \beta S(\theta) I(\theta) d\theta d\tau.$$

Here, we have used the equalities $x^1(\theta) = S(t + \theta)$ and $x^2(\theta) = I(t + \theta)$ for $\theta \leq 0$, and a linear shift $t + \theta \rightarrow \theta$ in the integral representation. Taking derivative with respect to t , we have from Equation (1)

$$\frac{d}{dt} [S(t) - S_0 \ln S(t)] = [S(t) - S_0] \left[\frac{b}{S(t)} - \beta I(t) - \mu_S + \frac{q\gamma I(t)}{S(t)} \right].$$

Making use of the identity $b = \mu_S S_0$ yields

$$\begin{aligned} \frac{b}{S(t)} - \beta I(t) - \mu_S + \frac{q\gamma I(t)}{S(t)} &= \frac{b}{S(t)} - \beta I(t) - \frac{b}{S_0} + \frac{q\gamma I(t)}{S(t)} - \frac{q\gamma I(t)}{S_0} + \frac{q\gamma I(t)}{S_0} \\ &= [b + q\gamma I(t)] \left[\frac{1}{S(t)} - \frac{1}{S_0} \right] - \left(\beta - \frac{q\gamma}{S_0} \right) I(t). \end{aligned}$$

Thus,

$$\frac{d}{dt} [S(t) - S_0 \ln S(t)] \leq - \left(\beta - \frac{q\gamma}{S_0} \right) [S(t) - S_0] I(t).$$

In view of Equation (2) and the definition of β_1 , we obtain

$$U'(t) \leq -\beta_1 [S(t) - S_0] I(t) + \beta_1 S(t) I(t) - (\mu_I + \gamma) I(t) = [\beta_1 S_0 - (\mu_I + \gamma)] I(t).$$

Since $R_0 \leq 1$, we have $\beta_1 S_0 \leq \mu_I + \gamma$ and consequently, $U'(t) \leq 0$. The largest invariant set of $U'(t) = 0$ is a singleton such that $S(t) \equiv S_0$ and $I(t) \equiv 0$. It follows from the Lyapunov–LaSalle invariance principle [8, p. 30] that the trivial equilibrium $(S_0, 0)$ is globally asymptotically stable if $R_0 \leq 1$.

For the case $R_0 > 1$, we construct the Lyapunov functional $V : X \rightarrow \mathbb{R}$ as

$$V(x^1, x^2) := \frac{\beta_1 S^*}{\beta S^* - q\gamma} V_S(x^1, x^2) + V_I(x^1, x^2) + V_-(x^1, x^2),$$

where

$$\begin{aligned} V_S(x^1, x^2) &:= x^1(0) - S^* \ln x^1(0), \\ V_I(x^1, x^2) &:= x^2(0) - I^* \ln x^2(0), \\ V_-(x^1, x^2) &:= \int_0^\infty \int_{-\tau}^0 p(\tau) e^{-\mu\tau} \beta [x^1(\theta)x^2(\theta) - S^* I^* \ln x^1(\theta)x^2(\theta)] d\theta d\tau. \end{aligned}$$

Restricting along a solution (S, I) of the system (1)–(2), we can rewrite V as

$$V(t) = \frac{\beta_1 S^*}{\beta S^* - q\gamma} V_S(t) + V_I(t) + V_-(t),$$

where

$$\begin{aligned} V_S(t) &= S(t) - S^* \ln S(t), \\ V_I(t) &= I(t) - I^* \ln I(t), \\ V_-(t) &= \int_0^\infty \int_{t-\tau}^t p(\tau) e^{-\mu\tau} \beta [S(\theta)I(\theta) - S^* I^* \ln S(\theta)I(\theta)] d\theta d\tau. \end{aligned}$$

Taking derivative with respect to t , we obtain from Equations (1) and (2)

$$\begin{aligned} V'_S(t) &= [S(t) - S^*] \left[\frac{b}{S(t)} - \beta I(t) - \mu_S + \frac{q\gamma I(t)}{S(t)} \right], \\ V'_I(t) &= [I(t) - I^*] \left[\int_0^\infty p(\tau) e^{-\mu\tau} \frac{\beta S(t-\tau)I(t-\tau)}{I(t)} d\tau - (\mu_I + \gamma) \right], \\ V'_-(t) &= \int_0^\infty p(\tau) e^{-\mu\tau} \beta \left[S(t)I(t) - S(t-\tau)I(t-\tau) + S^* I^* \ln \frac{S(t-\tau)I(t-\tau)}{S(t)I(t)} \right] d\tau. \end{aligned}$$

In view of $b - \beta S^* I^* - \mu_S S^* + q\gamma I^* = 0$, we have

$$\begin{aligned} \frac{b}{S(t)} - \beta I(t) - \mu_S + \frac{q\gamma I(t)}{S(t)} &= \frac{\beta S^* I^*}{S(t)} + \frac{\mu_S S^*}{S(t)} - \frac{q\gamma I^*}{S(t)} - \beta I(t) - \mu_S \\ &\quad + \frac{q\gamma I(t)}{S(t)} - \frac{q\gamma I(t)}{S^*} + \frac{q\gamma I(t)}{S^*} \\ &= [\mu_S S^* + q\gamma I(t)] \left[\frac{1}{S(t)} - \frac{1}{S^*} \right] + (\beta S^* - q\gamma) \left[\frac{I^*}{S(t)} - \frac{I(t)}{S^*} \right]. \end{aligned}$$

Therefore,

$$\frac{\beta_1 S^*}{\beta S^* - q\gamma} V'_S(t) \leq \beta_1 S^* [S(t) - S^*] \left[\frac{I^*}{S(t)} - \frac{I(t)}{S^*} \right].$$

On the other hand, since $\mu_I + \gamma = \beta_1 S^*$, we obtain from the definition of β_1 that

$$V'_I(t) = \int_0^\infty p(\tau) e^{-\mu\tau} \beta [I(t) - I^*] \left[\frac{S(t-\tau)I(t-\tau)}{I(t)} - S^* \right] d\tau.$$

Combining the above formulas and using the definition of β_1 , we have

$$V'(t) \leq \int_0^\infty p(\tau) e^{-\mu\tau} \beta W(t, \tau) d\tau,$$

where

$$\begin{aligned} W(t, \tau) := & S^* [S(t) - S^*] \left[\frac{I^*}{S(t)} - \frac{I(t)}{S^*} \right] + [I(t) - I^*] \left[\frac{S(t-\tau)I(t-\tau)}{I(t)} - S^* \right] \\ & + \left[S(t)I(t) - S(t-\tau)I(t-\tau) + S^* I^* \ln \frac{S(t-\tau)I(t-\tau)}{S(t)I(t)} \right]. \end{aligned}$$

Simplifying the above equation gives

$$W(t, \tau) = S^* I^* \left[2 - \frac{S^*}{S(t)} - \frac{S(t-\tau)I(t-\tau)}{S^* I(t)} + \ln \frac{S(t-\tau)I(t-\tau)}{S(t)I(t)} \right].$$

Note that $2 - a - b + \ln(ab) \leq 0$ for any $a > 0$ and $b > 0$; and the equality is satisfied if and only if $a = b = 1$. We obtain $W(t, \tau) \leq 0$ and consequently, $V'(t) \leq 0$. Moreover, the largest invariant set of $V'(t) = 0$ is a singleton where $S(t) \equiv S^*$ and $I(t) \equiv I^*$. By the Lyapunov–LaSalle invariance principle [8, p. 30], we obtain global asymptotic stability of the endemic equilibrium (S^*, I^*) under the condition $R_0 > 1$. ■

3. Discussion

We investigate a delayed SIS model and show that the global dynamics of this model depends on whether the basic reproductive ratio is greater than one. Our model is based on the assumption that the disease transmission occurs only when susceptible and infected individuals contact each other at the same time, namely, any susceptible individuals at present time cannot contact the infected individuals at the past time. This key assumption makes our model different from those considered in previous literature [3,4,6,9] in the sense that the incidence rate in our equation of susceptible individuals does not contain any delay. Thanks to this new assumption, we are able to establish a threshold theorem of global stability without any technical condition, while in the literature (see [3,4] for example), an additional assumption of sufficiently small probability of immunity lost is required to obtain global stability of endemic equilibrium. It is also noted that by choosing special distribution function $p(\tau)$ and using a standard linear chain trick [8, p. 96], our model reduces to the SEIS model with multiple latent classes considered in [2], and our result coincides with that obtained in [2].

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References

- [1] F.V. Atkinson and J.R. Haddock, *On determining phase spaces for functional differential equations*, Funkcial. Ekvac. 31 (1988), pp. 331–347.
- [2] N. Bame, S. Bowong, J. Mbang, G. Sallet, and J.-J. Tewa, *Global stability analysis for SEIS models with n latent classes* (English summary), Math. Biosci. Eng. 5(1) (2008), pp. 20–33.
- [3] Y. Enatsu, *Lyapunov functional techniques on the global stability of equilibria of SIS epidemic models with delays*, Kyoto Univ. Res. Inf. Repository 1792 (2012), pp. 118–130.
- [4] Y. Enatsu, Y. Nakata, and Y. Muroya, *Lyapunov functional techniques for the global stability analysis of a delayed SIRS epidemic model*, Nonlinear Anal. Real World Appl. 13 (2012), pp. 2120–2133.
- [5] J.K. Hale and S.M.V. Lunel, *Introduction to functional-differential equations*, Applied Mathematical Sciences, Vol. 99, Springer-Verlag, New York, 1993.
- [6] G. Huang and Y. Takeuchi, *Global analysis on delay epidemiological dynamics models with nonlinear incidence*, J. Math. Biol. 63 (2011), pp. 125–139.
- [7] A. Korobeinikov and P.K. Maini, *Nonlinear incidence and stability of infectious disease models*, Math. Med. Biol. 22 (2005), pp. 113–128.
- [8] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, San Diego, 1993.
- [9] C.C. McCluskey, *Complete global stability for an SIR epidemic model with delay-distributed or discrete*, Nonlinear Anal. Real World Appl. 11 (2010), pp. 55–59.