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# Uniform asymptotics of some $q$-orthogonal polynomials

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ABSTRACT

Uniform asymptotic formulas are obtained for the Stieltjes–Wigert polynomial, the $q^{-1}$-Hermite polynomial and the $q$-Laguerre polynomial as the degree of the polynomial tends to infinity. In these formulas, the $q$-Airy polynomial, defined by truncating the $q$-Airy function, plays a significant role. While the standard Airy function, used frequently in the uniform asymptotic formulas for classical orthogonal polynomials, behaves like the exponential function on one side and the trigonometric functions on the other side of an extreme zero, the $q$-Airy polynomial behaves like the $q$-Airy function on one side and the $q$-Theta function on the other side. The last two special functions are involved in the local asymptotic formulas of the $q$-orthogonal polynomials. It seems therefore reasonable to expect that the $q$-Airy polynomial will play an important role in the asymptotic theory of the $q$-orthogonal polynomials.



## 1. Introduction

For fixed $q \in (0,1)$, the $q$-shifted factorials [1, (1.2.15)] are defined by

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=1}^{n}\left(1-aq^{k-1}\right), \quad n=1,2,\ldots; \tag{1.1}$$

this definition remains valid when $n$ is infinite. We shall also make use of the identity

$$(q;q)_n = \frac{(q;q)_\infty}{(q^{n+1};q)_\infty}. \tag{1.2}$$

In terms of the notation in (1.1), the Stieltjes–Wigert polynomial [5, (3.27.1)] is given by

$$S_n(z;q) = \sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}(-z)^k. \tag{1.3}$$

In this paper, we are concerned with the asymptotic behavior of this polynomial as $n \to \infty$. First, let us introduce the new scale $z := q^{-nt}u$ with $u \in \mathbb{C}$, $u \neq 0$ and $t \in \mathbb{R}$. In view of the symmetry relation

$$S_n\left(q^{-nt}u;q\right) = (-u)^n q^{n^2(1-t)} S_n\left(q^{-n(2-t)}u^{-1}\right), \tag{1.4}$$

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we may restrict ourselves to the case $t \geqslant 1$; see [6, (5.6)]. The case $t = 2$ has been studied by Ismail [3, Theorem 2.5], who gave an asymptotic formula for this polynomial involving the $q$-Airy function

$$A_q(z) := \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}(-z)^k; \tag{1.5}$$

see also [2, Theorem 21.8.7]. For the cases $t \geqslant 2$ and $1 \leqslant t < 2$, Ismail and Zhang [4, Theorem 2.3] have derived asymptotic formulas in terms of the $q$-Airy function and the $q$-Theta function [7, p. 463]

$$\Theta_q(z) := \sum_{k=-\infty}^{\infty} q^{k^2} z^k, \tag{1.6}$$

respectively. The results in [4] have been improved in our earlier paper [6], where simpler formulas and sharper bounds for the error terms are given. However, none of the formulas obtained thus far holds uniformly in a neighborhood of $t = 2$, and our intention here is just to provide such a result. It turns out that, in stead of the $q$-Airy function given in (1.5), our formula involves the polynomial

$$A_{q,n}(z) := \sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k}(-z)^k. \tag{1.7}$$

Since this is simply the $n$-th partial sum of the $q$-Airy function, we call it the *$q$-Airy polynomial*. For convenience, we also introduce the *half $q$-Theta function*:

$$\Theta_q^+(z) := \sum_{k=0}^{\infty} q^{k^2} z^k. \tag{1.8}$$

Clearly,

$$\Theta_q(z) + 1 = \Theta_q^+(z) + \Theta_q^+(1/z). \tag{1.9}$$

The following theorem is our main result.

**Theorem 1.** *Let $z := q^{-nt}u$ with $t \geqslant 1$, $u \in \mathbb{C}$ and $|u| \geqslant 1/R$, where $R > 0$ is any fixed large number. Given any small $\delta > 0$, we have*

$$S_n(z;q) = \frac{(-z)^n q^{n^2}}{(q;q)_n}\left[A_{q,n}\left(q^{-2n}/z\right) + r_n(z)\right] \tag{1.10}$$

*for $t > 2(1-\delta)$, where the remainder satisfies*

$$\left|r_n(z)\right| \leqslant \frac{q^{n(1-3\delta)}}{1-q} A_{q,n}\left(-q^{-2n}/|z|\right) + \frac{q^{3n^2\delta^2 - 2n\delta} R^{\lfloor 3n\delta \rfloor + 1}}{(q;q)_\infty} \Theta_q^+\left(q^{4n\delta} R\right). \tag{1.11}$$

*In view of the symmetry relation* (1.4)*, the above result together with the second statement in Corollary* 2 *of* [6] *provide the asymptotic behavior of $S_n(z;q)$ for $z$ in the whole complex plane.*

The paper is arranged as follows. In Section 2 we present some asymptotic formulas for the $q$-Airy function and the $q$-Airy polynomial in terms of the $q$-Theta function. The proof of Theorem 1 is given in Section 3, where comparison of this result is also made with those in our earlier paper [6]. In Section 4 we state two theorems, corresponding to Theorem 1, for the $q^{-1}$-Hermite polynomial and the $q$-Laguerre polynomial.

## 2. Properties of the $q$-Airy function and the $q$-Airy polynomial

In this section, we have only one result, namely, the following.

**Proposition 1.** *Let $z := q^{-nt}u$ with $u \neq 0$ and $t$ being a fixed real number. When $t \geqslant 2$, we have*

$$A_{q,n}(z) = \frac{(-z)^n q^{n^2}}{(q;q)_\infty}\left[\Theta_q^+\left(-q^{-2n}/z\right) + O\left(q^{n(1-\delta)}\right)\right] \tag{2.1}$$

*uniformly for $|u| \geqslant 1/R$, where $\delta > 0$ is any small number and $R > 0$ is any large real number. When $0 < t < 2$, we have*

$$A_{q,n}(z) = \frac{(-z)^m q^{m^2}}{(q;q)_\infty}\left[\Theta_q\left(-q^{2m}z\right) + O\left(q^{m(1-\delta)}\right)\right], \tag{2.2}$$

*where $m := \lfloor \frac{nt}{2} \rfloor$ and $\delta > 0$ is any small number; this formula holds uniformly for $\frac{1}{R} \leqslant |u| \leqslant R$, where $R > 0$ is any large real number. When $t \leqslant 0$, we have*

$$A_{q,n}(z) = A_q(z) + O\left(q^{n^2(1-\delta)}\right) \tag{2.3}$$

*uniformly for $|u| \leqslant R$, where $\delta > 0$ is any small number. Furthermore, as $z \to \infty$, we have*

$$A_q(z) = \frac{(-z)^m q^{m^2}}{(q;q)_\infty}\left[\Theta_q\left(-q^{2m}z\right) + O\left(q^{m(1-\delta)}\right)\right], \tag{2.4}$$

*where $m := \lfloor \frac{\ln|z|}{-2\ln q} \rfloor$ and $\delta > 0$ is any small number.*

**Proof.** From the definition of $q$-Airy polynomial (1.7) we have

$$A_{q,n}(z) = \sum_{k=0}^{n} \frac{q^{(n-k)^2}}{(q;q)_{n-k}}(-z)^{n-k} = \frac{(-z)^n q^{n^2}}{(q;q)_\infty}\sum_{k=0}^{n} q^{k^2}\left(q^{n-k+1};q\right)_\infty\left(-q^{-2n}/z\right)^k.$$

If $t \geqslant 2$, we write

$$A_{q,n}(z) = \frac{(-z)^n q^{n^2}}{(q;q)_\infty}\left[\Theta_q^+\left(-q^{-2n}/z\right) + r_n(z)\right].$$

Then we have

$$r_n(z) = \sum_{k=0}^{n} q^{k^2}\left(q^{n-k+1};q\right)_\infty\left(-q^{-2n}/z\right)^k - \sum_{k=0}^{\infty} q^{k^2}\left(-q^{-2n}/z\right)^k = I_1 + I_2 + I_3,$$

where

$$I_1 := -\sum_{k=0}^{\lfloor n\delta\rfloor} q^{k^2}\left(1 - \left(q^{n-k+1};q\right)_\infty\right)\left(-q^{-2n}/z\right)^k,$$

$$I_2 := \sum_{k=\lfloor n\delta\rfloor+1}^{n} q^{k^2}\left(q^{n-k+1};q\right)_\infty\left(-q^{-2n}/z\right)^k,$$

$$I_3 := -\sum_{k=\lfloor n\delta\rfloor+1}^{\infty} q^{k^2}\left(-q^{-2n}/z\right)^k.$$

For any $0 \leqslant k \leqslant \lfloor n\delta\rfloor$, it is verifiable that

$$1 - \left(q^{n-k+1};q\right)_\infty < \frac{q^{n-k+1}}{1-q} < \frac{q^{n(1-\delta)}}{1-q}.$$

Since $|q^{-2n}/z| = q^{n(t-2)}/|u| \leqslant R$ for $t \geqslant 2$, we have

$$|I_1| \leqslant \frac{q^{n(1-\delta)}}{1-q}\sum_{k=0}^{\infty} q^{k^2}R^k = O\left(q^{n(1-\delta)}\right).$$

Furthermore, it is readily seen that

$$\max\left\{|I_2|, |I_3|\right\} \leqslant \sum_{k=\lfloor n\delta\rfloor+1}^{\infty} q^{k^2}R^k = \sum_{k=0}^{\infty} q^{(k+\lfloor n\delta\rfloor+1)^2}R^{k+\lfloor n\delta\rfloor+1}$$
$$\leqslant q^{n^2\delta^2}R^{n\delta+1}\Theta_q^+\left(q^{2n\delta}R\right) = O\left(q^{n^2\delta^2(1-\delta)}\right).$$

From the above estimates we obtain

$$A_{q,n}(z) = \frac{(-z)^n q^{n^2}}{(q;q)_\infty}\left[\Theta_q^+\left(-q^{-2n}/z\right) + O\left(q^{n(1-\delta)}\right)\right]$$

for any small $\delta > 0$. This proves (2.1).

Now we consider the case $0 < t < 2$. Set $m := \lfloor \frac{nt}{2} \rfloor$; then we can rewrite $q$-Airy polynomial as

$$A_{q,n}(z)=\sum_{k=0}^{n}\frac{q^{(k-m)^2-m^2}}{(q;q)_k}\bigl(-q^{2m}z\bigr)^k=\frac{(-z)^m q^{m^2}}{(q;q)_\infty}\sum_{k=-m}^{n-m}q^{k^2}\bigl(q^{k+m+1};q\bigr)_\infty\bigl(-q^{2m}z\bigr)^k. \tag{2.5}$$

To estimate the difference between the last sum and the $q$-Theta function, we let

$$\begin{aligned} r_n(z)&:=\frac{(q;q)_\infty}{(-z)^m q^{m^2}}A_{q,n}(z)-\Theta_q\bigl(-q^{2m}z\bigr)\\ &=\sum_{k=-m}^{n-m}q^{k^2}\bigl(q^{k+m+1};q\bigr)_\infty\bigl(-q^{2m}z\bigr)^k-\sum_{k=-\infty}^{\infty}q^{k^2}\bigl(-q^{2m}z\bigr)^k=I_1+I_2+I_3, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} I_1&:=\sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor}q^{k^2}\bigl(\bigl(q^{k+m+1};q\bigr)_\infty-1\bigr)\bigl(-q^{2m}z\bigr)^k,\\ I_2&:=\sum_{k=\lfloor n\delta\rfloor+1}^{n-m}q^{k^2}\bigl(q^{k+m+1};q\bigr)_\infty\bigl(-q^{2m}z\bigr)^k-\sum_{k=\lfloor n\delta\rfloor+1}^{\infty}q^{k^2}\bigl(-q^{2m}z\bigr)^k,\\ I_3&:=\sum_{k=-m}^{-\lfloor n\delta\rfloor-1}q^{k^2}\bigl(q^{k+m+1};q\bigr)_\infty\bigl(-q^{2m}z\bigr)^k-\sum_{k=-\infty}^{-\lfloor n\delta\rfloor-1}q^{k^2}\bigl(-q^{2m}z\bigr)^k. \end{aligned}$$

Firstly, since $1/R\leqslant|u|\leqslant R$ and $-2\leqslant 2m-nt\leqslant 0$, we have

$$q^2/R\leqslant\bigl|q^{2m}z\bigr|=q^{2m-nt}|u|\leqslant R/q^2.$$

Furthermore, for $-\lfloor n\delta\rfloor\leqslant k\leqslant\lfloor n\delta\rfloor$,

$$1-\bigl(q^{k+m+1};q\bigr)_\infty<\frac{q^{m-n\delta}}{1-q}.$$

Thus, it follows that

$$|I_1|\leqslant 2\frac{q^{m-n\delta}}{1-q}\sum_{k=0}^{\infty}q^{k^2}\bigl(R/q^2\bigr)^k=O\bigl(q^{m-n\delta}\bigr). \tag{2.7}$$

Secondly,

$$\begin{aligned} \max\bigl\{|I_2|,|I_3|\bigr\}&\leqslant 2\sum_{k=\lfloor n\delta\rfloor+1}^{\infty}q^{k^2}\bigl(R/q^2\bigr)^k=2\sum_{k=0}^{\infty}q^{(k+\lfloor n\delta\rfloor+1)^2}\bigl(R/q^2\bigr)^{k+\lfloor n\delta\rfloor+1}\\ &=2q^{(\lfloor n\delta\rfloor+1)^2}\bigl(R/q^2\bigr)^{\lfloor n\delta\rfloor+1}\Theta_q^+\bigl(q^{2\lfloor n\delta\rfloor}R\bigr)=O\bigl(q^{n^2\delta^2(1-\delta)}\bigr). \end{aligned} \tag{2.8}$$

Finally, applying the estimates (2.7) and (2.8) to (2.6), we obtain $r_n(z)=O(q^{m-n\delta})$. Therefore,

$$A_{q,n}(z)=\frac{(-z)^m q^{m^2}}{(q;q)_\infty}\bigl[\Theta_q\bigl(-q^{2m}z\bigr)+O\bigl(q^{m-n\delta}\bigr)\bigr]$$

for any small $\delta>0$. Replacing $\delta$ by $\frac{t\delta}{2}$, formula (2.2) then follows since $m:=\lfloor\frac{nt}{2}\rfloor$.

When $t\leqslant 0$, we have $q^{-nt}\leqslant 1$ and hence $|z|=|q^{-nt}u|\leqslant R$. From (1.5) and (1.7) we obtain

$$\bigl|A_{q,n}(z)-A_q(z)\bigr|=\left|\sum_{k=n+1}^{\infty}\frac{q^{k^2}}{(q;q)_k}(-z)^k\right|\leqslant\sum_{k=n+1}^{\infty}\frac{q^{k^2}}{(q;q)_\infty}R^k\leqslant\sum_{k=n}^{\infty}\frac{q^{k^2}}{(q;q)_\infty}R^k.$$

For convenience, we have added a positive term in the last sum. Since the last sum can be expressed in terms of the half $q$-Theta function defined in (1.8), we have

$$\bigl|A_{q,n}(z)-A_q(z)\bigr|\leqslant\sum_{l=0}^{\infty}\frac{q^{(l+n)^2}}{(q;q)_\infty}R^{l+n}=\frac{q^{n^2}R^n}{(q;q)_\infty}\Theta_q^+\bigl(q^{2n}R\bigr)=O\bigl(q^{n^2(1-\delta)}\bigr)$$

for any small $\delta>0$. This ends the proof of (2.3).

The proof of (2.4) is similar to that of (2.2). Recall that $m := \lfloor \frac{\ln|z|}{-2\ln q} \rfloor$. When $z$ tends to infinity, so does $m$. Furthermore, $1 \leqslant |q^{2m}z| \leqslant q^{-2}$. This suggests to change the variable in the $q$-Airy function from $z$ into $q^{2m}z$. On account of (1.2),

$$A_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}(-z)^k = \sum_{k=-m}^{\infty} \frac{q^{(k+m)^2}}{(q;q)_{k+m}}(-z)^{k+m}$$
$$= \frac{(-z)^m q^{m^2}}{(q;q)_\infty} \sum_{k=-m}^{\infty} q^{k^2}\left(q^{k+m+1};q\right)_\infty \left(-q^{2m}z\right)^k.$$

To prove (2.4) we only need to estimate the remainder

$$r(z) := \frac{(q;q)_\infty A_q(z)}{(-z)^m q^{m^2}} - \Theta_q\left(-q^{2m}z\right) = \sum_{k=-m}^{\infty} q^{k^2}\left(q^{k+m+1};q\right)_\infty \left(-q^{2m}z\right)^k - \sum_{k=-\infty}^{\infty} q^{k^2}\left(-q^{2m}z\right)^k$$
$$= I_1 + I_2 + I_3, \tag{2.9}$$

where

$$I_1 := \sum_{k=-\lfloor m\delta \rfloor}^{\lfloor m\delta \rfloor} q^{k^2}\left(\left(q^{k+m+1};q\right)_\infty - 1\right)\left(-q^{2m}z\right)^k,$$
$$I_2 := \sum_{k=\lfloor m\delta \rfloor+1}^{\infty} q^{k^2}\left(q^{k+m+1};q\right)_\infty \left(-q^{2m}z\right)^k - \sum_{k=\lfloor m\delta \rfloor+1}^{\infty} q^{k^2}\left(-q^{2m}z\right)^k,$$
$$I_3 := \sum_{k=-m}^{-\lfloor m\delta \rfloor-1} q^{k^2}\left(q^{k+m+1};q\right)_\infty \left(-q^{2m}z\right)^k - \sum_{k=-\infty}^{-\lfloor m\delta \rfloor-1} q^{k^2}\left(-q^{2m}z\right)^k.$$

Again since $q^2 < 1 \leqslant |q^{2m}z| \leqslant q^{-2}$, similarly to the proof of (2.2) one can show that for any fixed small $\delta > 0$,

$$|I_1| \leqslant 2\frac{q^{m-m\delta}}{1-q} \sum_{k=0}^{\infty} q^{k^2} q^{-2k} = O\left(q^{m(1-\delta)}\right), \tag{2.10}$$

and

$$\max\left\{|I_2|, |I_3|\right\} \leqslant 2 \sum_{k=\lfloor m\delta \rfloor+1}^{\infty} q^{k^2} q^{-2k} = 2q^{\lfloor m\delta \rfloor^2 - 1}\Theta_q^+\left(q^{2\lfloor m\delta \rfloor}\right) = O\left(q^{m^2\delta^2(1-\delta)}\right). \tag{2.11}$$

A combination of (2.9), (2.10) and (2.11) gives (2.4). This ends our proof. □

## 3. Proof of Theorem 1 and comparison with earlier results

From the definition of Stieltjes–Wigert polynomial (1.3), we have

$$S_n(z;q) = \sum_{k=0}^{n} \frac{q^{(n-k)^2}}{(q;q)_k (q;q)_{n-k}}(-z)^{n-k} = (-z)^n q^{n^2} \sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}}\left(-q^{-2n}/z\right)^k$$
$$= \frac{(-z)^n q^{n^2}}{(q;q)_n} \sum_{k=0}^{n} \frac{q^{k^2}(q^{n-k+1};q)_k}{(q;q)_k}\left(-q^{-2n}/z\right)^k. \tag{3.1}$$

Combining (1.10) with the definition of $q$-Airy polynomial (1.7) gives

$$r_n(z) := \frac{(q;q)_n}{(-z)^n q^{n^2}} S_n(z;q) - A_{q,n}\left(q^{-2n}/z\right) = -\sum_{k=0}^{n}\left(1 - \left(q^{n-k+1};q\right)_k\right)\frac{q^{k^2}}{(q;q)_k}\left(-q^{-2n}/z\right)^k.$$

We need to estimate $r_n(z)$ for $z = q^{-nt}u$ with $t > 2(1-\delta)$ and $|u| \geqslant 1/R$. Based on the idea of discrete Laplace's method [6], we divide the summation into two parts $\sum_{k=0}^{n} = \sum_{k=0}^{\lfloor n\delta_1 \rfloor} + \sum_{k=\lfloor n\delta_1 \rfloor+1}^{n}$ and estimate them separately. Here $\delta_1 \in (0,1)$ is a small number to be determined later. Put

$$r_n(z) = -I_1 - I_2 \tag{3.2}$$

with

$$I_1 := \sum_{k=0}^{\lfloor n\delta_1 \rfloor} \bigl(1 - \bigl(q^{n-k+1}; q\bigr)_k\bigr) \frac{q^{k^2}}{(q;q)_k} \bigl(-q^{-2n}/z\bigr)^k,$$

$$I_2 := \sum_{k=\lfloor n\delta_1 \rfloor+1}^{n} \bigl(1 - \bigl(q^{n-k+1}; q\bigr)_k\bigr) \frac{q^{k^2}}{(q;q)_k} \bigl(-q^{-2n}/z\bigr)^k.$$

In view of the inequality $1 - ab < (1-a) + (1-b)$ for any $a, b \in (0,1)$ and by induction, we can show that for any $0 \leqslant k \leqslant n\delta_1$,

$$1 - \bigl(q^{n-k+1}; q\bigr)_k < \sum_{i=1}^{k} q^{n-k+i} < \sum_{i=0}^{\infty} q^{n-k+i} = \frac{q^{n-k}}{1-q} \leqslant \frac{q^{n(1-\delta_1)}}{1-q}.$$

Thus, from the definition of $q$-Airy polynomial (1.7) we obtain

$$|I_1| \leqslant \sum_{k=0}^{\lfloor n\delta_1 \rfloor} \frac{q^{n(1-\delta_1)}}{1-q} \frac{q^{k^2}}{(q;q)_k} \bigl|q^{-2n}/z\bigr|^k \leqslant \frac{q^{n(1-\delta_1)}}{1-q} A_{q,n}\bigl(-q^{-2n}/|z|\bigr). \tag{3.3}$$

We can estimate $I_2$ by using the inequality $0 < 1 - (q^{n-k+1}; q)_k < 1$ for any $n\delta_1 \leqslant k \leqslant n$. Hence,

$$\begin{aligned} |I_2| &\leqslant \sum_{k=\lfloor n\delta_1 \rfloor+1}^{n} \frac{q^{k^2}}{(q;q)_\infty} \bigl|q^{-2n}/z\bigr|^k = \sum_{k=0}^{n-\lfloor n\delta_1 \rfloor-1} \frac{q^{(k+\lfloor n\delta_1 \rfloor+1)^2}}{(q;q)_\infty} \bigl|q^{-2n}/z\bigr|^{k+\lfloor n\delta_1 \rfloor+1} \\ &\leqslant \sum_{k=0}^{\infty} \frac{q^{(k+n\delta_1)^2}}{(q;q)_\infty} \bigl|q^{-2n}/z\bigr|^{k+\lfloor n\delta_1 \rfloor+1} = \frac{q^{n^2\delta_1^2} |q^{-2n}/z|^{\lfloor n\delta_1 \rfloor+1}}{(q;q)_\infty} \Theta_q^+\bigl(q^{2n\delta_1}\bigl|q^{-2n}/z\bigr|\bigr). \end{aligned}$$

Since $t > 2(1-\delta)$ and $|u| \geqslant 1/R$, we have $|q^{-2n}/z| = q^{-2n+nt}/|u| \leqslant q^{-2n\delta}R$. Therefore,

$$\begin{aligned} |I_2| &\leqslant \frac{q^{n^2\delta_1^2} q^{-2n\delta(\lfloor n\delta_1 \rfloor+1)} R^{\lfloor n\delta_1 \rfloor+1}}{(q;q)_\infty} \Theta_q^+\bigl(q^{2n(\delta_1-\delta)}R\bigr) \\ &\leqslant \frac{q^{n^2\delta_1^2 - 2n\delta(n\delta_1+1)} R^{\lfloor n\delta_1 \rfloor+1}}{(q;q)_\infty} \Theta_q^+\bigl(q^{2n(\delta_1-\delta)}R\bigr). \end{aligned} \tag{3.4}$$

Set $\delta_1 := 3\delta$. A combination of (3.2), (3.3) and (3.4) gives (1.11) immediately.

Now we compare Theorem 1 with our earlier results in [6, Corollary 2]. Recall $z := q^{-nt}u$.

1. We first consider the case $t \geqslant 2$ and $|u| \geqslant 1/R$, where $R > 0$ is a fixed large number. From (1.10) we have

$$\begin{aligned} S_n(z;q) &= \frac{(-z)^n q^{n^2}}{(q;q)_n} \bigl[A_{q,n}\bigl(q^{-2n}/z\bigr) + r_n(z)\bigr] \\ &= \frac{(-u)^n q^{n^2(1-t)}}{(q;q)_n} \bigl[A_q\bigl(u^{-1}q^{n(t-2)}\bigr) + A_{q,n}\bigl(u^{-1}q^{n(t-2)}\bigr) - A_q\bigl(u^{-1}q^{n(t-2)}\bigr) + r_n(z)\bigr]. \end{aligned} \tag{3.5}$$

Given any small $\delta > 0$, we wish to show that

$$\bigl|A_{q,n}\bigl(u^{-1}q^{n(t-2)}\bigr) - A_q\bigl(u^{-1}q^{n(t-2)}\bigr) + r_n(z)\bigr| = O\bigl(q^{n(1-3\delta)}\bigr).$$

Firstly, since $t \geqslant 2$ and $|u| \geqslant 1/R$ we have $|u^{-1}q^{n(t-2)}| \leqslant R$. From the definition of the $q$-Airy function (1.5) and the $q$-Airy polynomial (1.7) we obtain

$$\begin{aligned} \bigl|A_{q,n}\bigl(u^{-1}q^{n(t-2)}\bigr) - A_q\bigl(u^{-1}q^{n(t-2)}\bigr)\bigr| &= \left| \sum_{k=n+1}^{\infty} \frac{q^{k^2}}{(q;q)_k} \bigl(-u^{-1}q^{n(t-2)}\bigr)^k \right| \\ &\leqslant \sum_{k=n}^{\infty} \frac{q^{k^2}}{(q;q)_\infty} R^k = \sum_{l=0}^{\infty} \frac{q^{(l+n)^2}}{(q;q)_\infty} R^{l+n} = \frac{q^{n^2} R^n}{(q;q)_\infty} \Theta_q^+\bigl(q^{2n}R\bigr) \\ &= O\bigl(q^{n^2(1-\delta)}\bigr). \end{aligned} \tag{3.6}$$

Here for the sake of simplicity we have used $\sum_{k=n}^{\infty}$ in stead of $\sum_{k=n+1}^{\infty}$ on the right of the inequality.

Secondly, since $|z| = q^{-nt}|u| \geqslant q^{-2n}/R$, we have from (1.11) that

$$\begin{aligned}\left|r_n(z)\right| &\leqslant \frac{q^{n(1-3\delta)}}{1-q} A_{q,n}\left(-q^{-2n}/|z|\right) + \frac{q^{3n^2\delta^2-2n\delta} R^{\lfloor 3n\delta\rfloor+1}}{(q;q)_\infty} \Theta_q^+\left(q^{4n\delta}R\right) \\ &\leqslant \frac{q^{n(1-3\delta)}}{1-q} A_q(-R) + O\left(q^{3n^2\delta^2(1-\delta)}\right) = O\left(q^{n(1-3\delta)}\right). \end{aligned} \tag{3.7}$$

Coupling (3.6) and (3.7), we obtain from (3.5)

$$S_n(z;q) = \frac{(-u)^n q^{n^2(1-t)}}{(q;q)_n}\left[A_q\left(u^{-1}q^{n(t-2)}\right) + O\left(q^{n(1-3\delta)}\right)\right],$$

which coincides with [6, (5.15)].

2. Now we consider the case $2(1-\delta) < t < 2$ and $1/R \leqslant |u| \leqslant R$, where $R > 0$ and $\delta \in (0, 1/4)$ are fixed. From (1.10) and (1.11), it follows

$$S_n(z;q) = \frac{(-u)^n q^{n^2(1-t)}}{(q;q)_n}\left[A_{q,n}\left(u^{-1}q^{-n(2-t)}\right) + r_n(z)\right], \tag{3.8}$$

where

$$\left|r_n(z)\right| \leqslant \frac{q^{n(1-3\delta)}}{1-q} A_{q,n}\left(-|u|^{-1}q^{-n(2-t)}\right) + \frac{q^{3n^2\delta^2-2n\delta} R^{\lfloor 3n\delta\rfloor+1}}{(q;q)_\infty} \Theta_q^+\left(q^{4n\delta}R\right). \tag{3.9}$$

Set $m := \lfloor \frac{n(2-t)}{2} \rfloor$. Since $1/R \leqslant |u| \leqslant R$ and $0 < 2-t < 2\delta < 2$, the conditions of (2.2) in Proposition 1 are satisfied with $t$ replaced by $2-t$. Thus we obtain the asymptotic formulas

$$A_{q,n}\left(u^{-1}q^{-n(2-t)}\right) = \frac{(-u^{-1}q^{-n(2-t)})^m q^{m^2}}{(q;q)_\infty}\left[\Theta_q\left(-u^{-1}q^{2m-n(2-t)}\right) + O\left(q^{m(1-\delta)}\right)\right], \tag{3.10}$$

$$A_{q,n}\left(-|u|^{-1}q^{-n(2-t)}\right) = \frac{(|u|^{-1}q^{-n(2-t)})^m q^{m^2}}{(q;q)_\infty}\left[\Theta_q\left(|u|^{-1}q^{2m-n(2-t)}\right) + O\left(q^{m(1-\delta)}\right)\right]. \tag{3.11}$$

Applying (3.10) to (3.8) gives

$$S_n(z;q) = \frac{(-u)^{n-m} q^{n^2(1-t)-m[n(2-t)-m]}}{(q;q)_n(q;q)_\infty}\left[\Theta_q\left(-u^{-1}q^{2m-n(2-t)}\right) + O\left(q^{m(1-\delta)}\right) + \frac{(q;q)_\infty r_n(z)}{(-u^{-1}q^{-n(2-t)})^m q^{m^2}}\right], \tag{3.12}$$

and applying (3.11) to (3.9) yields

$$\begin{aligned}\left|\frac{(q;q)_\infty r_n(z)}{(-u^{-1}q^{-n(2-t)})^m q^{m^2}}\right| &\leqslant \frac{q^{n(1-3\delta)}}{1-q}\left[\Theta_q\left(|u|^{-1}q^{2m-n(2-t)}\right) + O\left(q^{m(1-\delta)}\right)\right] + \frac{q^{3n^2\delta^2-2n\delta} R^{\lfloor 3n\delta\rfloor+1}}{(|u|^{-1}q^{-n(2-t)})^m q^{m^2}} \Theta_q^+\left(q^{4n\delta}R\right) \\ &= O\left(q^{n(1-3\delta)}\right) + O\left(q^{n(1-3\delta)+m(1-\delta)}\right) + O\left(q^{3n^2\delta^2-2n\delta+m^2} R^{3n\delta+m}\right) = O\left(q^{m(1-\delta)}\right), \end{aligned} \tag{3.13}$$

where we have used $q^{n(1-3\delta)} < q^m < q^{m(1-\delta)}$ by observing that $2(1-\delta) < t < 2$ with $\delta \in (0, 1/4)$ implies $m := \lfloor \frac{n(2-t)}{2} \rfloor \leqslant \lfloor n\delta \rfloor < n(1-3\delta)$. Finally, coupling (3.12) and (3.13), we have

$$S_n(z;q) = \frac{(-u)^{n-m} q^{n^2(1-t)-m[n(2-t)-m]}}{(q;q)_n(q;q)_\infty}\left[\Theta_q\left(-u^{-1}q^{2m-n(2-t)}\right) + O\left(q^{m(1-\delta)}\right)\right], \tag{3.14}$$

which agrees with [6, (5.16)].

## 4. The $q^{-1}$-Hermite polynomial and the $q$-Laguerre polynomial

The $q^{-1}$-Hermite polynomial [2, (21.2.5)] is defined by

$$h_n(\sinh\xi\,|q) := \sum_{k=0}^{n} \frac{(q^{n-k+1};q)_k}{(q;q)_k} q^{k^2-nk}(-1)^k e^{(n-2k)\xi}. \tag{4.1}$$

We set $z = \sinh\xi_n := \frac{1}{2}(q^{-nt}u - q^{nt}u^{-1})$ with $u \in \mathbb{C}$, $u \neq 0$ and $t \geqslant 0$. In [6, Corollary 1] we have derived asymptotic formulas for $t \geqslant 1/2$ and $0 \leqslant t < 1/2$ respectively. Here we will give an asymptotic formula which holds uniformly for $t$ in a neighborhood of 1/2. Our result involves the $q$-Airy polynomial defined in (1.7).

**Theorem 2.** *Let $z = \sinh \xi_n := \frac{1}{2}(q^{-nt}u - q^{nt}u^{-1})$ with $u \in \mathbb{C}$ and $|u| \geqslant 1/R$, where $R > 0$ is any fixed large number. Given any small $\delta > 0$, we have*

$$h_n(\sinh \xi_n | q) = u^n q^{-n^2 t}\big[A_{q,n}\big(u^{-2}q^{n(2t-1)}\big) + r_n(t,u)\big] \tag{4.2}$$

*for $t > 1/2 - \delta$, where the remainder satisfies*

$$\big|r_n(t,u)\big| \leqslant \frac{q^{n(1-3\delta)}}{1-q} A_{q,n}\big(-|u|^{-2}q^{n(2t-1)}\big) + \frac{q^{3n^2\delta^2 - 2n\delta} R^{2(\lfloor 3n\delta \rfloor + 1)}}{(q;q)_\infty} \Theta_q^+\big(q^{4n\delta}R^2\big). \tag{4.3}$$

**Proof.** Since $e^{\xi_n} = q^{-nt}u$, from (4.1), (4.2) and the definition of the $q$-Airy polynomial in (1.7) it is easy to see that

$$r_n(t,u) = \sum_{k=0}^{n} \frac{(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}\big(-u^{-2}q^{n(2t-1)}\big)^k = I_1 + I_2, \tag{4.4}$$

where

$$I_1 := \sum_{k=0}^{\lfloor n\delta_1 \rfloor} \frac{(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}\big(-u^{-2}q^{n(2t-1)}\big)^k,$$

$$I_2 := \sum_{k=\lfloor n\delta_1 \rfloor + 1}^{n} \frac{(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}\big(-u^{-2}q^{n(2t-1)}\big)^k.$$

Here $\delta_1 \in (0,1)$ is a small number to be determined later. For any $0 \leqslant k \leqslant \lfloor n\delta_1 \rfloor$,

$$0 \leqslant 1 - \big(q^{n-k+1};q\big)_k < \frac{q^{n-k+1}}{1-q} \leqslant \frac{q^{n(1-\delta_1)}}{1-q}.$$

Thus,

$$|I_1| \leqslant \sum_{k=0}^{n} \frac{q^{n(1-\delta_1)}}{1-q} \frac{q^{k^2}}{(q;q)_k}\big(|u|^{-2}q^{n(2t-1)}\big)^k = \frac{q^{n(1-\delta_1)}}{1-q} A_{q,n}\big(-|u|^{-2}q^{n(2t-1)}\big). \tag{4.5}$$

Furthermore, since $0 \leqslant 1 - (q^{n-k+1};q)_k \leqslant 1$ for any $k \in [0,n]$ and $|u|^{-2}q^{n(2t-1)} \leqslant q^{-2n\delta}R^2$ for $t > 1/2 - \delta$, we obtain

$$\begin{aligned} |I_2| &\leqslant \sum_{k=\lfloor n\delta_1 \rfloor + 1}^{\infty} \frac{q^{k^2}}{(q;q)_\infty}\big(q^{-2n\delta}R^2\big)^k \\ &= \frac{q^{(\lfloor n\delta_1 \rfloor + 1)^2 - 2n\delta(\lfloor n\delta_1 \rfloor + 1)} R^{2(\lfloor n\delta_1 \rfloor + 1)}}{(q;q)_\infty} \Theta_q^+\big(q^{2(\lfloor n\delta_1 \rfloor + 1) - 2n\delta}R^2\big) \\ &\leqslant \frac{q^{n^2\delta_1^2 - 2n\delta(n\delta_1 + 1)} R^{2(\lfloor n\delta_1 \rfloor + 1)}}{(q;q)_\infty} \Theta_q^+\big(q^{2n\delta_1 - 2n\delta}R^2\big). \end{aligned} \tag{4.6}$$

Choose $\delta_1 := 3\delta$. Then (4.3) follows from (4.4), (4.5) and (4.6). □

The $q$-Laguerre polynomial [2, (21.8.1)] is defined by

$$L_n^\alpha(z) := \sum_{k=0}^{n} \frac{(q^{\alpha+k+1};q)_{n-k}}{(q;q)_k (q;q)_{n-k}} q^{k^2+\alpha k}(-z)^k. \tag{4.7}$$

Let $z := q^{-nt}u$ with $u \in \mathbb{C}$, $u \neq 0$ and $t \geqslant 1$. In [6, Corollary 3] we gave two asymptotic formulas, one for $t \geqslant 2$ and the other for $1 \leqslant t < 2$. Here we will use the $q$-Airy polynomial defined in (1.7) to derive an asymptotic formula holding uniformly in a neighborhood of $t = 2$.

**Theorem 3.** *Assume that $\alpha$ is real and $\alpha > -1$. Let $z := q^{-nt}u$ with $u \in \mathbb{C}$ and $|u| \geqslant 1/R$, where $R > 0$ is any fixed large number. Given any small $\delta > 0$, we have*

$$L_n^\alpha(z) = \frac{(-zq^\alpha)^n q^{n^2}}{(q;q)_n}\big[A_{q,n}\big(q^{-2n-\alpha}/z\big) + r_n(z)\big] \tag{4.8}$$

*for $t > 2(1-\delta)$, where the remainder satisfies*

$$\left|r_n(z)\right| \leqslant \frac{2q^{n(1-2\delta)}}{1-q} A_{q,n}\left(-q^{-2n-\alpha}/|z|\right) + \frac{q^{3n^2\delta^2-2n\delta}(q^{-\alpha}R)^{\lfloor 3n\delta\rfloor+1}}{(q;q)_\infty} \Theta_q^+\left(q^{4n\delta-\alpha}R\right). \tag{4.9}$$

**Proof.** From the definition of $q$-Laguerre polynomial (4.7) we have

$$\begin{aligned} L_n^\alpha(z) &= \sum_{k=0}^{n} \frac{(q^{\alpha+n-k+1};q)_k}{(q;q)_k(q;q)_{n-k}} q^{(n-k)^2+\alpha(n-k)}(-z)^{n-k} \\ &= \frac{(-zq^\alpha)^n q^{n^2}}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{\alpha+n-k+1};q)_k(q^{n-k+1};q)_k}{(q;q)_k} q^{k^2}\left(-q^{-2n-\alpha}/z\right)^k. \end{aligned} \tag{4.10}$$

Thus, it follows from (4.10) and the definition of $q$-Airy polynomial (1.7) that the remainder in (4.8) can be written as

$$r_n(z) = \sum_{k=0}^{n} \frac{(q^{\alpha+n-k+1};q)_k(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}\left(-q^{-2n-\alpha}/z\right)^k = I_1 + I_2, \tag{4.11}$$

where

$$I_1 := \sum_{k=0}^{\lfloor n\delta_1\rfloor} \frac{(q^{\alpha+n-k+1};q)_k(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}\left(-q^{-2n-\alpha}/z\right)^k,$$

$$I_2 := \sum_{k=\lfloor n\delta_1\rfloor+1}^{n} \frac{(q^{\alpha+n-k+1};q)_k(q^{n-k+1};q)_k - 1}{(q;q)_k} q^{k^2}\left(-q^{-2n-\alpha}/z\right)^k.$$

Here $\delta_1 \in (0,1)$ is a small number to be determined later. Since $\alpha > -1$, for any $0 \leqslant k \leqslant \lfloor n\delta_1\rfloor$ we have

$$0 \leqslant 1 - \left(q^{\alpha+n-k+1};q\right)_k\left(q^{n-k+1};q\right)_k < \frac{q^{\alpha+n-k+1}+q^{n-k+1}}{1-q} < \frac{2q^{n(1-\delta_1)}}{1-q}.$$

Therefore,

$$|I_1| \leqslant \sum_{k=0}^{n} \frac{2q^{n(1-\delta_1)}}{1-q} \frac{q^{k^2}}{(q;q)_k}\left(q^{-2n-\alpha}/|z|\right)^k = \frac{2q^{n(1-\delta_1)}}{1-q} A_{q,n}\left(-q^{-2n-\alpha}/|z|\right). \tag{4.12}$$

Also, since $0 \leqslant 1 - (q^{\alpha+n-k+1};q)_k(q^{n-k+1};q)_k \leqslant 1$ for any $k \in [0,n]$ and $q^{-2n-\alpha}/|z| \leqslant q^{-2n\delta-\alpha}R$ for $t > 2(1-\delta)$, we obtain

$$\begin{aligned} |I_2| &\leqslant \sum_{k=\lfloor n\delta_1\rfloor+1}^{\infty} \frac{q^{k^2}}{(q;q)_\infty}\left(q^{-2n\delta-\alpha}R\right)^k \\ &= \frac{q^{(\lfloor n\delta_1\rfloor+1)^2-2n\delta(\lfloor n\delta_1\rfloor+1)}(q^{-\alpha}R)^{\lfloor n\delta_1\rfloor+1}}{(q;q)_\infty} \Theta_q^+\left(q^{2(\lfloor n\delta_1\rfloor+1)-2n\delta-\alpha}R\right) \\ &\leqslant \frac{q^{n^2\delta_1^2-2n\delta(n\delta_1+1)}(q^{-\alpha}R)^{\lfloor n\delta_1\rfloor+1}}{(q;q)_\infty} \Theta_q^+\left(q^{2n\delta_1-2n\delta-\alpha}R\right). \end{aligned} \tag{4.13}$$

Set $\delta_1 := 3\delta$, then (4.9) follows from (4.11), (4.12) and (4.13). □

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