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From continuous to discrete: weak limit of normalized Askey–Wilson measure

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Abstract
In this paper, we consider the weak limit of the normalized measure for Askey–Wilson polynomials when the parameter q approaches -1 from the right. We use two different methods to prove that the weak limit is a discrete measure with two mass points that are symmetric about the origin. The weights on these two mass points are, however, not always the same. We also calculate the weak limit of the q-ultraspherical measure when q approaches a complex root of unity.

Keywords Askey–Wilson polynomials · Orthogonality measures · Weak limit

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1 Introduction

Given two infinite sequences $\alpha_n \in \mathbb{R}$ and $\beta_n > 0$, we consider the following second-order linear difference equation:

$$y_{n+1}(x) = (x - \alpha_n) y_n(x) - \beta_n y_{n-1}(x), \quad n \geq 1, \tag{1.1}$$

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which has two linearly independent solutions $\{p_n(x)\}$ and $\{p_n^*(x)\}$ with initial conditions

$$p_0(x) = 1, \quad p_1(x) = x - \alpha_0, \quad p_0^*(x) = 0, \quad p_1^*(x) = 1. \tag{1.2}$$

It is readily seen that $p_n(x)$ and $p_n^*(x)$ are monic polynomials in x with degrees n and $n-1$, respectively. Moreover, applying the difference equation (1.1) recursively gives us

$$p_{n+1}^*(x)p_n(x) - p_{n+1}(x)p_n^*(x) = \beta_1 \dots \beta_n. \tag{1.3}$$

By the spectral theorem for orthogonal polynomials [20, Theorem 2.5.2], $\{p_n(x)\}_{n=1}^{\infty}$ are orthogonal polynomials with respect to some probability measure $d\mu(x)$ on $\mathbb{R}$. Moreover, for any $N \in \mathbb{N}$, all the zeros of $p_N(x)$ are real and simple. We denote them as $z_1, \dots, z_N$. It follows from Christoffel-Darboux formula [20, Theorem 2.2.2], [28, Theorem 3.2.2] that

$$\sum_{k=0}^{N-1} \frac{p_k(z_r)p_k(z_s)}{\beta_1\beta_2 \dots \beta_k} = \frac{p_N'(z_r)p_{N-1}(z_r)}{\beta_1\beta_2 \dots \beta_{N-1}}\delta_{r,s},$$

which, in view of (1.3), is the same as

$$\sum_{k=0}^{N-1} \frac{p_k(z_r)p_k(z_s)p_N^*(z_r)}{\beta_1\beta_2 \dots \beta_k p_N'(z_r)} = \delta_{r,s}.$$

Here, $\delta_{r,s}$ denotes the Kronecker delta function which equals one when $r = s$ and zero when $r \neq s$. Define two $N \times N$ matrices A and B as $A_{rj} = p_{j-1}(z_r)p_N^*(z_r)/p_N'(z_r)$ and $B_{js} = p_{j-1}(z_s)/(\beta_1 \dots \beta_{j-1})$. The above equation is equivalent to $AB = I$, where I is the identity matrix. It then follows from $BA = I$ that

$$\sum_{r=1}^{N} \frac{p_j(z_r)p_k(z_r)p_N^*(z_r)}{\beta_1\beta_2 \dots \beta_j p_N'(z_r)} = \delta_{j,k}, \qquad \text{for} \quad 0 \le j, k \le N-1.$$

This implies that the polynomials $p_k(x)$ with $k = 0, \dots, N-1$ are orthogonal with respect to the discrete measure

$$d\mu_N(x) = \sum_{r=1}^{N} \frac{p_N^*(z_r)}{p_N'(z_r)}\delta(x - z_r),$$

where $\delta(x)$ is the Dirac delta measure at 0, noting that $\delta(x)$ and $\delta_{r,s}$ share the same symbol δ but have different meanings. It is readily seen from the above formula that

$$\int_{\mathbb{R}} \frac{d\mu_N(x)}{z - x} = \frac{p_N^*(z)}{p_N(z)}, \tag{1.4}$$

for $z \in \mathbb{C}$ such that $p_N(z) \neq 0$. If $d\mu(x)$ is compactly supported, we let $N \to \infty$ in the above equation and obtain Markov's theorem; see [20, Theorem 2.6.2]. Note that $p_N^*(z_r)/p_N'(z_r)$ is also the Gaussian quadrature coefficient; namely, if $f \in \mathbb{P}_{2N-1}(x)$ is any polynomial of degree no more than $2N-1$, then the Gaussian quadrature formula is exact:

$$\int_{\mathbb{R}} f(x)d\mu(x) = \sum_{r=1}^{N} \frac{p_N^*(z_r)}{p_N'(z_r)} f(z_r) = \int_{\mathbb{R}} f(x)d\mu_N(x).$$

Since $p_N^*(z)$ and $p_N(z)$ are monic polynomials with degree $N-1$ and N, respectively, we have $\int_{\mathbb{R}} d\mu_N(x) = 1$. Moreover, the interlacing property of the zeros of $p_N(z)$ and $p_{N-1}(z)$ implies that $p_{N-1}(z_r)p_N'(z_r) > 0$. By (1.3), $p_N^*(z_r)p_{N-1}(z_r) = \beta_1 \ldots \beta_{N-1}$ is also positive. Hence, $d\mu_N(x)$ is a discrete probability measure, and the orthogonality can be written as follows:

$$\int_{\mathbb{R}} p_j(x)p_k(x)d\mu_N(x) = \beta_1\beta_2\ldots\beta_j\delta_{j,k}, \quad \text{for } j,k = 0,1,\ldots,N-1. \tag{1.5}$$

Now, we assume that the coefficients α_n and β_n in (1.1) depend continuously on a parameter q such that $\alpha_n(q) \in \mathbb{R}$ and $\beta_n(q) > 0$ for all $n > 0$ and $q \in (c,d)$. Assume that the limits of $\alpha_n(q)$ and $\beta_n(q)$ as q tend to c from the right exist, and there exists $N > 1$ such that $\beta_N(c) = 0$ and $\beta_n(c) > 0$ for all $n = 0,\ldots,N-1$. We note that there are many cases in the literature when β_n becomes zero for some n and certain related choices of the parameters. This leads to the so-called isochronous systems [13]. Many related papers are [8–12, 14–16, 22]. There is a major difference between the limiting measure and the measure for the so-called -1-polynomials in the literature [18, 29, 30]. For example, it was assumed in [30] that the other parameters also vary along with q so that the recurrence coefficients $\beta_n(-1)$ remain positive for all $n \geq 0$; see [30, (2.22)]. However, in our assumption, the recurrence coefficients $\beta_n(-1)$ vanish for some finite n, and hence, the three-term recurrence relation is degenerate.

One motivation of the present paper comes from an investigation of the continuous q-Hermite polynomials $H_n(x\,|\,q)$, $0 < q < 1$, which are orthogonal with respect to the following weight:

$$w_H(x\,|\,q) = \frac{(q, e^{2i\theta}, e^{-2i\theta}; q)_\infty}{2\pi\sqrt{1-x^2}}, \qquad x \in [-1,1] \tag{1.6}$$

with $x = \cos\theta$; see [20, (13.1.11)–(13.1.12)]. They satisfy a three-term recurrence relation as follows:

$$H_{n+1}(x\,|\,q) = 2xH_n(x\,|\,q) - (1-q^n)H_{n-1}(x\,|\,q); \tag{1.7}$$

see [20, (13.1.1)]. Note that $H_n(x\,|\,q) = (2x)^n +$ lower-order terms. Comparing the above formula with (1.1), the recurrence coefficients are given by

$$\alpha_n(q) = 0, \qquad \beta_n(q) = \frac{1-q^n}{4}.$$

Both coefficients are continuous for all $n > 0$ and $q \in (-1, 1)$, and $\beta_2(-1) = 0$.

In the literature, Szablowski [27] observed that the density function $w_H(x \mid q)$ in (1.6) is unimodal and looks like a normal distribution when $q > 0$. He also observed that it becomes bimodal if $-1 < q < q_0$, where $q_0 \simeq -0.107$ is the largest real root of the equation

$$\sum_{k=0}^{\infty}(2k+1)^2 q^{k(k+1)/2} = 0.$$

This interesting work was followed by Deng and Yang [17] who accurately graphed $w_H(x \mid q)$ when q is close to -1. Based on numerical results, they conjectured that, as $q \to -1$, $w_H(x \mid q)$ tends to a sum of two discrete masses at $\pm\sqrt{2}/2$ with weights $1/2$ at each point.

In this paper, we will show that Deng and Yang's conjecture is true. In fact, we will prove a theorem for a large class of orthogonal polynomials. This class includes the Askey–Wilson polynomials $p_n(x; \mathbf{t} \mid q)$, which are considered as the most general orthogonal polynomials in the q-Askey scheme.

The rest of the paper is organized as follows. In Sect. 2, we prove our main result under a general assumption on the recurrence coefficients $\alpha_n(q)$ and $\beta_n(q)$. We establish a similar result for the Askey–Wilson polynomials in Sect. 3. Since these polynomials satisfy some explicit identities, a different approach will be presented. Then, as special cases of the Askey–Wilson polynomials, more examples in the q-Askey scheme are considered in Sect. 4, including a complex case in Sect. 5. We finish the paper with a discussion in Sect. 6. Throughout this paper, we follow the notation and terminology in the books [4, 19, 20], as well as the Askey scheme [23].

2 The main theorem

Theorem 2.1 *Let $\alpha_n(q)$ with $n \geq 0$ and $\beta_n(q)$ with $n \geq 1$ be continuous functions of $q \in [c, d)$ such that $\alpha_n(q) \in \mathbb{R}$ and $\beta_n(q) > 0$ for all $q \in (c, d)$. Let $p_n(x|q)$ and $p_n^*(x|q)$ be monic polynomials satisfying the following second-order linear difference equation:*

$$y_{n+1}(x|q) = [x - \alpha_n(q)]y_n(x|q) - \beta_n(q)y_{n-1}(x|q) \tag{2.1}$$

with initial conditions

$$p_0(x|q) = 1, \;\; p_1(x|q) = x - \alpha_0(q), \quad p_0^*(x|q) = 0, \;\; p_1^*(x|q) = 1.$$

Assume that for each $q \in (c, d)$, $d\mu(x|q)$ is a compactly supported and normalized positive measure for $p_n(x|q)$; i.e., there exists a compact interval $S \subset \mathbb{R}$ independent of q such that

$$\int_S p_j(x|q) d\mu(x|q) = \delta_{j,0}, \quad j = 0, 1, \ldots .$$

If there exists $N > 1$ such that $\beta_N(c) = 0$ and $\beta_n(c) > 0$ for all $n = 1, \ldots, N-1$, then, as $q \to c^+$, the measure $d\mu(x|q)$ converges weakly to a discrete probability measure $d\mu_N(x|c)$; namely, for any continuous function $f \in C(S)$, we have

$$\lim_{q\to c^+} \int_S f(x)d\mu(x|q) = \int_S f(x)d\mu_N(x|c) = \sum_{r=1}^{N} \frac{f(z_r(c))p_N^*(z_r(c)|c)}{p_N'(z_r(c)|c)}, \quad (2.2)$$

where $z_1(c), \ldots, z_N(c)$ are zeros of $p_N(x|c)$.

Proof Let $z_1(q), \ldots, z_N(q) \in S$ be the zeros of $p_N(x|q)$. By (1.5), we have

$$\int_S p_j(x|c)p_k(x|c)d\mu_N(x|c) = \beta_1(c)\beta_2(c)\ldots\beta_j(c)\delta_{j,k} \qquad \text{for} \quad 0 \le j,k \le N-1.$$

Under the assumption $\beta_N(c) = 0$, we obtain from the recurrence relation (2.1) that $p_{N+1}(z_r(c)|c) = (z_r(c) - \alpha_N)p_N(z_r(c)|c) = 0$. This gives us $p_j(z_r(c)|c) = 0$ for all $j \ge N$. Consequently, we have

$$\int_S p_j(x|c)d\mu_N(x|c) = \delta_{j,0}, \quad \text{for all } j \ge 0.$$

Recall our assumption that $d\mu(x|q)$ is the normalized measure for the orthogonal polynomials $p_j(x|q)$ such that $\int_S p_j(x|q)d\mu(x|q) = \delta_{j,0}$ for all $j \ge 0$. As this integral is actually q-independent, we may compare it with the above formula and put it in the following form:

$$\lim_{q\to c^+} \int_S p_j(x|q)d\mu(x|q) = \int_S p_j(x|c)d\mu_N(x|c), \quad \text{for all } j \ge 0.$$

Next, let us expand x^n in terms of $p_k(x|q)$ as follows:

$$x^n = \sum_{k=0}^{n} \lambda_{n,k}(q)p_k(x|q).$$

Note that the above coefficients $\lambda_{n,k}(q)$ are (finite) multiple sums of products of the $\alpha_j(q)\beta_{j+1}(q)$ with $0 \le j \le n$. It is obvious that $\lambda_{n,k}(q) \to \lambda_{n,k}(c)$ as $q \to c^+$, which gives us

$$\int_S x^n d\mu(x|q) = \lambda_{n,0}(q) \to \lambda_{n,0}(c) = \int_S x^n d\mu_N(x|c)$$

as $q \to c^+$. On account of the above limit relation, for any polynomial $P_n(x)$, there exists $q_\varepsilon > c$ such that

$$\left|\int_S P_n(x)d\mu(x|q) - \int_S P_n(x)d\mu_N(x|c)\right| < \varepsilon$$

for all $q \in (c, q_\varepsilon)$. Note that we have assumed that the measures $d\mu(x|q)$ have uniform compact support S. By Weierstrass approximation theorem, given any $\varepsilon > 0$ and a continuous function $f(x)$ on S, there exists a polynomial $P_n(x)$ such that $|f(x) - P_n(x)| < \varepsilon$ for all $x \in S$. Since

$$\left|\int_S f(x)d\mu(x|q) - \int_S P_n(x)d\mu(x|q)\right| \leq \max_{x\in S} |f(x) - P_n(x)| \int_S d\mu(x|q) = \varepsilon$$

for all $q \in (c, d)$, and

$$\left|\int_S f(x)d\mu_N(x|c) - \int_S P_n(x)d\mu_N(x|c)\right| \leq \max_{x\in S} |f(x) - P_n(x)| \int_S d\mu_N(x|c) = \varepsilon,$$

we have

$$\left|\int_S f(x)d\mu(x|q) - \int_S f(x)d\mu_N(x|c)\right| < 3\varepsilon$$

for all $q \in (c, q_\varepsilon)$. This implies that

$$\lim_{q\to c^+} \int_S f(x)d\mu(x|q) = \int_S f(x)d\mu_N(x|c)$$

for any continuous function $f(x)$, which finishes the proof of the theorem. □

Remark 2.2 If $N = 2$ in Theorem 2.1, we have $p_2^*(x|c) = x - \alpha_1(c)$ and $p_2(x|c) = [x - \alpha_1(c)][x - \alpha_0(c)] - \beta_1(c)$. Then, the measure $d\mu_2(x|c)$ in (2.2) is given explicitly as follows:

$$d\mu_2(x|c) = \frac{z_1(c) - \alpha_1(c)}{z_1(c) - z_2(c)} d\delta(x - z_1(c)) + \frac{z_2(c) - \alpha_1(c)}{z_2(c) - z_1(c)} d\delta(x - z_2(c)),$$

where $z_1(c) + z_2(c) = \alpha_1(c) + \alpha_0(c)$ and $z_1(c)z_2(c) = \alpha_1(c)\alpha_0(c) - \beta_1(c)$.

In the following sections, we will study the Askey–Wilson polynomials and their special cases as concrete examples.

3 The Askey–Wilson polynomials

Listed at the top of the q-Askey scheme, the Askey–Wilson polynomials $p_n(x; \mathbf{t}\,|\,q)$ are considered as the most general q-orthogonal polynomials. With $x = \cos\theta$ and the parameter $\mathbf{t} = (t_1, t_2, t_3, t_4)$, they are defined as follows:

$$p_n(x; \mathbf{t}\,|\,q) = t_1^{-n}(t_1t_2, t_1t_3, t_1t_4; q)_n\, {}_4\phi_3\left(\left.{q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1e^{i\theta}, t_1e^{-i\theta} \atop t_1t_2, t_1t_3, t_1t_4}\right| q, q\right); \quad (3.1)$$

see [20, (15.2.5)]. One may rewrite it in a form as follows:

$$\frac{p_n(x;\mathbf{t}\,|\,q)}{(q,t_1t_2,t_3t_4;q)_n}=\sum_{k=0}^{n}\frac{(t_1e^{i\theta},t_2e^{i\theta};q)_k(t_3e^{-i\theta},t_4e^{-i\theta};q)_{n-k}}{(q,t_1t_2;q)_k(q,t_3t_4;q)_{n-k}}e^{i(n-2k)\theta};$$

see [20, (15.2.8)]. When $\max|t_i|<1$, $i=1,\dots,4$, they satisfy the following orthogonality relation:

$$\int_{-1}^{1}p_m(x;\mathbf{t}\,|\,q)p_n(x;\mathbf{t}\,|\,q)w(x,\mathbf{t}\,|\,q)\mathrm{d}x$$
$$=\frac{(q;q)_n(t_1t_2t_3t_4q^{n-1};q)_n\prod_{1\le j<k\le 4}(t_jt_k;q)_n}{(t_1t_2t_3t_4;q)_{2n}}\delta_{m,n},\tag{3.2}$$

where the weight function is given by

$$w(x,\mathbf{t}\,|\,q)$$
$$=\frac{(q,e^{2i\theta},e^{-2i\theta};q)_\infty\prod_{1\le j<k\le 4}(t_jt_k;q)_\infty}{(t_1t_2t_3t_4;q)_\infty\prod_{j=1}^{4}(t_je^{i\theta},t_je^{-i\theta};q)_\infty}\frac{1}{2\pi\sqrt{1-x^2}}\tag{3.3}$$

with $x=\cos\theta$; see [20, (15.2.4)]. Note that, we have normalized the above weight function such that

$$\int_{-1}^{1}w(x,\mathbf{t}\,|\,q)\mathrm{d}x=1.$$

For the Askey–Wilson polynomials with different parameters

$$\mathbf{a}=(a_1,a_2,a_3,a_4)\quad\text{and}\quad\mathbf{b}=(b_1,b_2,b_3,b_4),$$

it is well known that they satisfy the following connection relation (cf. [20, Theorem 16.4.2]):

$$p_n(x;\mathbf{b}\,|\,q)$$
$$=\sum_{k=0}^{n}\frac{p_k(x;\mathbf{a}\,|\,q)(q;q)_n(b_1b_2b_3b_4q^{n-1};q)_k(b_1b_4,b_2b_4,b_3b_4;q)_n}{(q;q)_{n-k}(q,a_1a_2a_3a_4q^{k-1};q)_k(b_1b_4,b_2b_4,b_3b_4;q)_kq^{k(n-k)}b_4^{n-k}}$$
$$\times\sum_{j,l\ge 0}$$
$$\frac{(q^{k-n},b_1b_2b_3b_4q^{n+k-1},a_4b_4q^k;q)_{j+l}(a_1a_4q^k,a_2a_4q^k,a_3a_4q^k;q)_l(b_4/a_4;q)_jq^{j+l}}{(b_1b_4q^k,b_2b_4q^k,b_3b_4q^k;q)_{j+l}(q;q)_j(q;q)_l(a_4b_4q^k,a_1a_2a_3a_4q^{2k};q)_l(a_4/b_4)^l}.\tag{3.4}$$

Due to the relations among the Askey–Wilson polynomials and the other polynomials in the Askey scheme, it is possible to choose some special parameters $\mathbf{b}$ and replace the right-hand side of the above formula with other ones. In other words, certain polynomials in the Askey scheme can be expanded in terms of the Askey–Wilson polynomials $p_n(x; \mathbf{a} \,|\, q)$ with general parameter $\mathbf{a}$, where the coefficients are given explicitly. Indeed, we have the following result for the Chebyshev polynomials.

Lemma 3.1 *Let*

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}, \qquad x = \cos\theta,$$

be the Chebyshev polynomials of the second kind. Then, we have

$$\begin{aligned} U_n(x) = \sum_{k=0}^{n} &\frac{p_k(x; \mathbf{a} \,|\, q)(q; q)_n (q^{n+2}; q)_k (-q^{3/2}, q^{3/2}, -q^2; q)_n}{(q^{n+2}; q)_n (q; q)_{n-k} (q, a_1a_2a_3a_4q^{k-1}; q)_k (-q^{3/2}, q^{3/2}, -q^2; q)_k q^{k(n-k)} (-q)^{n-k}} \\ \times \sum_{j,l \geq 0} &\frac{(q^{k-n}, q^{n+k+2}, -a_4q^{k+1}; q)_{j+l} (a_1a_4q^k, a_2a_4q^k, a_3a_4q^k; q)_l (-q/a_4; q)_j q^{j+l}}{(-q^{3/2+k}, q^{3/2+k}, -q^{2+k}; q)_{j+l} (q; q)_j (q; q)_l (-a_4q^{k+1}, a_1a_2a_3a_4q^{2k}; q)_l (-a_4/q)^l}. \end{aligned} \tag{3.5}$$

Proof The q-ultraspherical polynomials $C_n(x; \beta \,|\, q)$ are special cases of the Askey–Wilson polynomials (cf. [20, (15.2.15)]):

$$p_n(x; \sqrt{\beta}, -\sqrt{\beta}, \sqrt{\beta q}, -\sqrt{\beta q} \,|\, q) = \frac{(q, -\beta; q)_n (q\beta^2; q^2)_n}{(\beta^2; q)_n} C_n(x; \beta \,|\, q).$$

Note that $U_n(x) = C_n(x; q \,|\, q)$ (cf. [20, (13.2.14)]). Then, we have from the above formula:

$$U_n(x) = \frac{p_n(x; \sqrt{q}, -\sqrt{q}, q, -q \,|\, q)}{(q^{n+2}; q)_n}. \tag{3.6}$$

Therefore, by choosing $b_1 = \sqrt{q}$, $b_2 = -\sqrt{q}$, $b_3 = q$ and $b_4 = -q$, we obtain (3.5) from (3.4) and (3.6). □

Next, we integrate both sides of (3.5) with respect to the normalized Askey–Wilson weight (3.3). Due to the orthogonality relation (3.2), all terms vanish on the right-hand side except for the term when $k = 0$. This gives us

$$
\begin{aligned}
&\int_{-1/\lambda_q}^{1/\lambda_q} U_n(\lambda_q x) w(\lambda_q x, \mathbf{a} \,|\, q) \lambda_q \mathrm{d}x \\
&= \int_{-1}^{1} U_n(x) w(x, \mathbf{a} \,|\, q) \mathrm{d}x = \frac{(q^3; q^2)_n (-q^2; q)_n}{(-q)^n (q^{n+2}; q)_n} \\
&\quad \times \sum_{j,l \geq 0} \frac{(q^{-n}, q^{n+2}; q)_{j+l} (a_1 a_4, a_2 a_4, a_3 a_4; q)_l (-q/a_4, -q^{l+1} a_4; q)_j q^{j+l}}{(q^3; q^2)_{j+l} (-q^2; q)_{j+l} (q; q)_j (q; q)_l (a_1 a_2 a_3 a_4; q)_l (-a_4/q)^l},
\end{aligned}
\tag{3.7}
$$

where λ_q is a continuous function of q in a small interval $[-1, -1+\sigma]$ with $\lambda_{-1} = 1$ and $0 < \lambda_q < 1$ for $q \in (-1, -1+\sigma)$. A typical example is $\lambda_q = \sqrt{(1-q)/2}$. Next, we will compute the limit of the above formula as $q \to -1$. To facilitate our future computations, let us list a few useful relations as follows:

$$
(q^n; q)_k \sim 2^k \left(\frac{n}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor} (1+q)^{\lfloor \frac{k+1}{2} \rfloor}, \qquad \text{if } n \text{ is even,} \tag{3.8}
$$

$$
(q^n; q)_k \sim 2^k \left(\frac{n+1}{2}\right)_{\lfloor \frac{k}{2} \rfloor} (1+q)^{\lfloor \frac{k}{2} \rfloor}, \qquad \text{if } n \text{ is odd,} \tag{3.9}
$$

$$
(-q^2; q)_k \sim 2^k \left(\frac{3}{2}\right)_{\lfloor \frac{k}{2} \rfloor} (1+q)^{\lfloor \frac{k}{2} \rfloor}, \tag{3.10}
$$

as $q \to -1$. Then, we establish the following lemmas.

Lemma 3.2 *For any integer $n \geq 0$, we have*

$$
\lim_{q \to -1} \frac{(q^3; q^2)_n (-q^2; q)_n}{(-q)^n (q^{n+2}; q)_n} = n+1. \tag{3.11}
$$

Proof Obviously, $(q^3; q^2)_n \to 2^n$ and $(-q)^n \to 1$ as $q \to -1$. When $n = 2m$ is even, we have from (3.8) and (3.10)

$$
\frac{(-q^2; q)_n}{(q^{n+2}; q)_n} \sim \frac{2^n (3/2)_m (1+q)^m}{2^n (m+1)_m (1+q)^m} = \frac{(1)_m (3/2)_m}{(1)_m (m+1)_m} = \frac{(2)_n}{2^n (1)_n} = \frac{n+1}{2^n}
$$

as $q \to -1$. Similarly, when $n = 2m+1$ is odd, we have from (3.10)

$$
\frac{(-q^2; q)_n}{(q^{n+2}; q)_n} \sim \frac{2^n (3/2)_m (1+q)^m}{2^n (m+2)_m (1+q)^m} = \frac{(1)_{m+1} (3/2)_m}{(1)_{m+1} (m+2)_m} = \frac{(2)_n}{2^n (1)_n} = \frac{n+1}{2^n}
$$

as $q \to -1$. A simple calculation gives (3.11). □

Lemma 3.3 *Denote*

$$
A = \frac{(1-a_4^2)(1-1/a_4^2)}{4}, \quad B = \frac{(1-a_1^2 a_4^2)(1-a_2^2 a_4^2)(1-a_3^2 a_4^2)}{4a_4^2 (1-a_1^2 a_2^2 a_3^2 a_4^2)}.
$$

We have

$$\lim_{q\to -1}\frac{(a_1a_4,a_2a_4,a_3a_4;q)_l(-q/a_4,-q^{l+1}a_4;q)_jq^{j+l}}{(q^3;q^2)_{j+l}(a_1a_2a_3a_4;q)_l(-a_4/q)^l}$$
$$=\begin{cases} A^iB^k, & j=2i, l=2k,\\ -\frac{(1-a_1a_4)(1-a_2a_4)(1-a_3a_4)}{2a_4(1-a_1a_2a_3a_4)}A^iB^k, & j=2i, l=2k+1,\\ -\frac{(1-a_4)(1-1/a_4)}{2}A^iB^k, & j=2i+1, l=2k,\\ \frac{(1-a_1a_4)(1-a_2a_4)(1-a_3a_4)(1+a_4)(1-1/a_4)}{4a_4(1-a_1a_2a_3a_4)}A^iB^k, & j=2i+1, l=2k+1. \end{cases} \tag{3.12}$$

Proof The proof is based on the repeated uses of

$$\lim_{q\to -1}(c;q)_j=\begin{cases}(1-c^2)^i, & j=2i,\\ (1-c^2)^i(1-c), & j=2i+1,\end{cases}$$

for $c^2\neq 1$. □

Lemma 3.4 *If $n=2m$ is even, we have*

$$\lim_{q\to -1}\frac{(q^{-n},q^{n+2};q)_{j+l}}{(-q^2;q)_{j+l}(q;q)_j(q;q)_l}=\begin{cases}\frac{(-m)_{i+k}(m+1)_{i+k}}{(3/2)_{i+k}(1)_i(1)_k} & j=2i, l=2k,\\ 0 & otherwise.\end{cases} \tag{3.13}$$

If $n=2m+1$ is odd, we have

$$\lim_{q\to -1}\frac{(q^{-n},q^{n+2};q)_{j+l}}{(-q^2;q)_{j+l}(q;q)_j(q;q)_l}=\begin{cases}0 & j=2i+1, l=2k+1,\\ \frac{(-m)_{i+k}(m+2)_{i+k}}{(3/2)_{i+k}(1)_i(1)_k} & otherwise,\end{cases} \tag{3.14}$$

where $i=\lfloor j/2\rfloor$ and $k=\lfloor l/2\rfloor$.

Proof For convenience, let us denote

$$C_{j,l}=\frac{(q^{-n},q^{n+2};q)_{j+l}}{(-q^2;q)_{j+l}(q;q)_j(q;q)_l}.$$

We first consider the case when $n=2m$ is even. From (3.8)–(3.10), we have

$$C_{j,l}=O\left((1+q)^{\lfloor (j+l+1)/2\rfloor+\lfloor (j+l+1)/2\rfloor-\lfloor (j+l)/2\rfloor-\lfloor j/2\rfloor-\lfloor l/2\rfloor}\right),\quad \text{as}\quad q\to -1,$$

which implies that $C_{j,l}\to 0$ if either j or l is odd. When both $j=2i$ and $l=2k$ are even, we obtain

$$C_{j,l}\sim\frac{2^{2i+2k}(-m)_{i+k}(1+q)^{i+k}2^{2i+2k}(m+1)_{i+k}(1+q)^{i+k}}{2^{2i+2k}(3/2)_{i+k}(1+q)^{i+k}2^{2i}(1)_i(1+q)^i2^{2k}(1)_k(1+q)^k}=\frac{(-m)_{i+k}(m+1)_{i+k}}{(3/2)_{i+k}(1)_i(1)_k},$$

as $q \to -1$. Then, we obtain (3.13).

Next, we consider the case when $n = 2m + 1$ is odd. Again, from (3.8)–(3.10), we have

$$C_{j,l} = O\left((1+q)^{\lfloor (j+l)/2 \rfloor + \lfloor (j+l)/2 \rfloor - \lfloor (j+l)/2 \rfloor - \lfloor j/2 \rfloor - \lfloor l/2 \rfloor}\right), \quad \text{as} \quad q \to -1,$$

which implies that $C_{j,l} \to 0$ if both j and l are odd. There are three cases to be considered. (i) When $j = 2i$ and $l = 2k$, we obtain

$$C_{j,l} \sim \frac{2^{2i+2k}(-m)_{i+k}(1+q)^{i+k} 2^{2i+2k}(m+2)_{i+k}(1+q)^{i+k}}{2^{2i+2k}(3/2)_{i+k}(1+q)^{i+k} 2^{2i}(1)_i(1+q)^i 2^{2k}(1)_k(1+q)^k} = \frac{(-m)_{i+k}(m+2)_{i+k}}{(3/2)_{i+k}(1)_i(1)_k}.$$

(ii) When $j = 2i$ and $l = 2k + 1$, we have

$$\begin{aligned} C_{j,l} &\sim \frac{2^{2i+2k+1}(-m)_{i+k}(1+q)^{i+k} 2^{2i+2k+1}(m+2)_{i+k}(1+q)^{i+k}}{2^{2i+2k+1}(3/2)_{i+k}(1+q)^{i+k} 2^{2i}(1)_i(1+q)^i 2^{2k+1}(1)_k(1+q)^k} \\ &= \frac{(-m)_{i+k}(m+2)_{i+k}}{(3/2)_{i+k}(1)_i(1)_k}. \end{aligned}$$

(iii) When $j = 2i + 1$ and $l = 2k$, we obtain

$$\begin{aligned} C_{j,l} &\sim \frac{2^{2i+2k+1}(-m)_{i+k}(1+q)^{i+k} 2^{2i+2k+1}(m+2)_{i+k}(1+q)^{i+k}}{2^{2i+2k+1}(3/2)_{i+k}(1+q)^{i+k} 2^{2i+1}(1)_i(1+q)^i 2^{2k}(1)_k(1+q)^k} \\ &= \frac{(-m)_{i+k}(m+2)_{i+k}}{(3/2)_{i+k}(1)_i(1)_k}. \end{aligned}$$

Finally, the above four formulas give us (3.14). □

With the above lemmas, we are ready to evaluate the limit of (3.7) as $q \to -1$. When $n = 2m$ is even, a combination of (3.7), (3.11), (3.12), and (3.13) yields

$$\begin{aligned} &\lim_{q \to -1} \int_{-1/\lambda_q}^{1/\lambda_q} U_{2m}(\lambda_q x) w(\lambda_q x, a_1, a_2, a_3, a_4 \,|\, q) \lambda_q \mathrm{d}x \\ &= (2m+1) \sum_{i,k \ge 0} \frac{(-m)_{i+k}(m+1)_{i+k}}{(3/2)_{i+k}(1)_i(1)_k} A^i B^k \\ &= (2m+1) \sum_{s=0}^{m} \frac{(-m)_s(m+1)_s}{(3/2)_s(1)_s} (A+B)^s. \end{aligned} \tag{3.15}$$

Note that, the series on the right-hand side is indeed the polynomial U_{2m} itself. To see it, from (18.7.4) and (18.7.13) in [24], we have

$$U_{2m}(x) = (2m+1) \frac{P_m^{(\frac{1}{2},-\frac{1}{2})}(2x^2-1)}{P_m^{(\frac{1}{2},-\frac{1}{2})}(1)},$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomials. From the series representation of $P_n^{(\alpha,\beta)}(x)$ in [24, Eq.(18.5.7)], we have from the above formula

$$U_{2m}(x) = (2m+1)\sum_{s=0}^{m}\frac{(-m)_s(m+1)_s}{(3/2)_s(1)_s}(1-x^2)^s. \tag{3.16}$$

By setting

$$x_0 = \sqrt{1-A-B} = \sqrt{\frac{2-2a_1^2a_2^2a_3^2a_4^2+\sum_{j=1}^4 a_j^2-\sum_{1\le i<j<k\le 4}a_i^2a_j^2a_k^2}{4(1-a_1^2a_2^2a_3^2a_4^2)}}, \tag{3.17}$$

we have from (3.15) and (3.16)

$$\lim_{q\to -1}\int_{-1/\lambda_q}^{1/\lambda_q} U_{2m}(\lambda_q x)w(\lambda_q x, a_1, a_2, a_3, a_4\,|\,q)\lambda_q \mathrm{d}x = U_{2m}(x_0). \tag{3.18}$$

Here, $x_0>0$ because $1-a_1^2a_2^2a_3^2a_4^2>0$ and

$$\sum_{j=1}^4 a_j^2 - \sum_{1\le i<j<k\le 4} a_i^2a_j^2a_k^2 = (a_1^2+a_2^2)(1-a_3^2a_4^2)+(a_3^2+a_4^2)(1-a_1^2a_2^2)>0$$

for all $a_1, a_2, a_3, a_4 \in (-1,1)$. Similarly, when $n=2m+1$ is odd, it follows from (3.7), (3.11), (3.12) and (3.14) that

$$\begin{aligned}
&\lim_{q\to -1}\int_{-1/\lambda_q}^{1/\lambda_q} U_{2m+1}(\lambda_q x)w(\lambda_q x, a_1, a_2, a_3, a_4\,|\,q)\lambda_q \mathrm{d}x\\
&\quad = (2m+2)y_0\sum_{i,k\ge 0}\frac{(-m)_{i+k}(m+2)_{i+k}}{(3/2)_{i+k}(1)_i(1)_k}A^iB^k\\
&\quad = (2m+2)y_0\sum_{s=0}^{m}\frac{(-m)_s(m+2)_s}{(3/2)_s(1)_s}(A+B)^s.
\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}
y_0 &:= 1-\frac{(1-a_1a_4)(1-a_2a_4)(1-a_3a_4)}{2a_4(1-a_1a_2a_3a_4)}-\frac{(1-a_4)(1-1/a_4)}{2}\\
&= \frac{\sum_{j=1}^4 a_j - \sum_{1\le i<j<k\le 4}a_ia_ja_k}{2(1-a_1a_2a_3a_4)}.
\end{aligned} \tag{3.20}$$

Again, the right-hand side of (3.19) is also the polynomial U_{2m+1}. Similar to the derivation of (3.16), we have from (18.5.7), (18.7.4), and (18.7.14) in [24] that

$$U_{2m+1}(x) = (2m+2)x \sum_{s=0}^{m} \frac{(-m)_s(m+2)_s}{(3/2)_s(1)_s}(1-x^2)^s. \tag{3.21}$$

Then, the above formula and (3.19) yield

$$\lim_{q\to -1} \int_{-1/\lambda_q}^{1/\lambda_q} U_{2m+1}(\lambda_q x) w(\lambda_q x, a_1, a_2, a_3, a_4 \,|\, q)\lambda_q \mathrm{d}x = \frac{y_0}{x_0} U_{2m+1}(x_0). \tag{3.22}$$

Since U_n is an even function when n is even, and is an odd function when n is odd. One can unify (3.18) and (3.22) as the following single equation:

$$\lim_{q\to -1} \int_{-1/\lambda_q}^{1/\lambda_q} U_n(\lambda_q x) w(\lambda_q x, a_1, a_2, a_3, a_4 \,|\, q)\lambda_q \mathrm{d}x$$
$$= \frac{x_0+y_0}{2x_0} U_n(x_0) + \tfrac{x_0-y_0}{2x_0} U_n(-x_0). \tag{3.23}$$

Finally, the above result gives us a theorem for the Askey–Wilson weight as follows.

Theorem 3.5 *Given* $\mathbf{a} = (a_1, a_2, a_3, a_4)$ *with* $a_1, a_2, a_3, a_4 \in (-1, 1)$*, and let* $w(x, \mathbf{a} \,|\, q)$ *be the weight function of the Askey–Wilson polynomials defined in* (3.3)*. Let* x_0 *and* y_0 *be the constants defined in* (3.17) *and* (3.20)*, respectively. Then, we have*

$$\lim_{q\to -1} \int_{-1/\lambda_q}^{1/\lambda_q} f(\lambda_q x) w(\lambda_q x, \mathbf{a} \,|\, q)\lambda_q \mathrm{d}x = \frac{x_0+y_0}{2x_0} f(x_0) + \frac{x_0-y_0}{2x_0} f(-x_0), \tag{3.24}$$

for any $f \in C[-1, 1]$.

Proof For simplicity, let us denote

$$J_q(h) := \int_{-1/\lambda_q}^{1/\lambda_q} h(\lambda_q x) w(\lambda_q x, a_1, a_2, a_3, a_4 \,|\, q)\lambda_q \mathrm{d}x,$$
$$Q(h) := \frac{x_0+y_0}{2x_0} h(x_0) + \frac{x_0-y_0}{2x_0} h(-x_0),$$

for any $h \in C[-1, 1]$. By Weierstrass approximation theorem, for any $\varepsilon > 0$, there exist an integer $N > 0$ and a polynomial $P_N(x)$ of degree N, such that $|f(x) - P_N(x)| < \varepsilon$ for all $x \in [-1, 1]$. This gives us $|Q(f) - Q(P_N)| < \varepsilon$ for any $f \in C[-1, 1]$. Note that P_N can be expressed as a linear combination of $U_0, \ldots, U_N$. It then follows from (3.23) that there exists $q_0 \in (-1,1)$ such that $|J_q(P_N) - Q(P_N)| < \varepsilon$ for all $q \in (-1, q_0)$. Since $w(x, a_1, a_2, a_3, a_4 \,|\, q) > 0$ for all $x \in (-1,1)$ and $q \in (-1,1)$, we have $|J_q(f) - J_q(P_N)| < \varepsilon$ for all $q \in (-1,1)$. By triangular inequalities, we obtain

$$|J_q(f) - Q(f)| \le |J_q(f) - J_q(P_N)| + |J_q(P_N) - Q(P_N)| + |Q(P_N) - Q(f)| < 3\varepsilon$$

for all $q \in (-1, q_0)$. This completes the proof. □

Corollary 3.6 *Given $\mathbf{a} = (a_1, a_2, a_3, a_4)$ with $a_1, a_2, a_3, a_4 \in (-1,1)$, and let $w(x, \mathbf{a} \,|\, q)$ be the weight function of the Askey–Wilson polynomials defined in (3.3). We have*

$$w(\lambda_q x, \mathbf{a} \,|\, q)\mathrm{d}x \rightharpoonup \frac{x_0 + y_0}{2x_0}\delta(x - x_0) + \frac{x_0 - y_0}{2x_0}\delta(x + x_0),$$

as $q \to -1^+$, where x_0 and y_0 are the constants defined in (3.17) and (3.20), respectively. From here onwards, the notation "$\rightharpoonup$" represents the convergence of the measure in a weak sense.

Proof For any $f \in C[-1,1]$, we can extend it to a continuous and bounded function on the whole real line by defining $f(x) = f(-1)$ for $x < -1$ and $f(x) = f(1)$ for $x > 1$. Since $\lambda_q \to 1$ and $f(x) - f(\lambda_q x) \to 0$ as $q \to -1$, we have

$$\int_{-1}^{1} f(x) w(\lambda_q x, \mathbf{a} \,|\, q)\mathrm{d}x - \int_{-1}^{1} f(\lambda_q x) w(\lambda_q x, \mathbf{a} \,|\, q)\lambda_q \mathrm{d}x \to 0$$

as $q \to -1$. Moreover, the integrals of $|f(\lambda_q x) w(\lambda_q x, \mathbf{a} \,|\, q)|$ on the shrinking intervals $[-1/\lambda_q, -1]$ and $[1, 1/\lambda_q]$ also vanish as $q \to -1$. Hence, we obtain from (3.24) that

$$\lim_{q \to -1} \int_{-1}^{1} f(x) w(\lambda_q x, \mathbf{a} \,|\, q)\mathrm{d}x = \frac{x_0 + y_0}{2x_0} f(x_0) + \frac{x_0 - y_0}{2x_0} f(-x_0).$$

This completes the proof. □

Remark 3.7 Given $a_1, a_2, a_3, a_4 \in (-1,1)$, the monic Askey–Wilson polynomials satisfy the second-order linear difference equation:

$$p_{n+1}(x|q) = [x - B_n(q)/2] p_n(x|q) - A_{n-1}(q) C_n(q) p_{n-1}(x|q),$$

where

$$A_n = \frac{1 - a_1a_2a_3a_4q^{n-1}}{(1 - a_1a_2a_3a_4q^{2n-1})(1 - a_1a_2a_3a_4q^{2n})},$$
$$C_n = \frac{(1 - q^n)\prod_{1\le j<k\le 4}(1 - a_ja_kq^{n-1})}{(1 - a_1a_2a_3a_4q^{2n-2})(1 - a_1a_2a_3a_4q^{2n-1})},$$
$$B_n = a_1 + \frac{1}{a_1} - \frac{A_n}{a_1}\prod_{j=2}^{4}(1 - a_1a_jq^n) - \frac{a_1C_n}{\prod_{j=2}^{4}(1 - a_1a_jq^{n-1})};$$

see [20, (15.2.10)–(15.2.13)]. It is readily seen that $C_2(-1) = 0$, which implies $c = -1$ and $N = 2$ in Theorem 2.1. A further calculation gives $\alpha_0(-1) = y_0$, $\alpha_1(-1) = -y_0$, and $\beta_1(-1) = x_0^2 - y_0^2$, where x_0 and y_0 are defined in (3.17) and (3.20), respectively. Then, a direct application of Theorem 2.1 and Remark 2.2 also gives (3.24). We also remark that $\pm x_0$ are the zeros of second-order Askey–Wilson polynomials $p_2(x\,|-1)$; see [26, (2.6)]. Note that the definitions of the Askey–Wilson polynomials are slightly different in the present paper and [26]. Namely, we have $p_n(x|q) = C_nP_n(2x)$ for certain constant C_n and the Askey–Wilson polynomial $P_n(x)$ defined in [26, (1.9)].

4 Some special cases

Since the Askey–Wilson polynomials are the most general orthogonal polynomials in the q-Askey scheme, they can be reduced to other polynomials by choosing certain parameters $\mathbf{t}$ in their definition in (3.1).

4.1 The Al-Salam–hihara polynomials

The Al-Salam–Chihara polynomials were first introduced in [3]. They appeared in a characterization theorem describing all orthogonal polynomials satisfying a certain convolution property. Al-Salam and Chihara identified the recurrence relation satisfied by their polynomials and derived a generating function. The weight function was found in [6], and later they were identified as special Askey–Wilson polynomials in [7]. Indeed, by setting $t_3 = t_4 = 0$ in (3.1), the Al-Salam–Chihara polynomials $Q_n(x; t_1, t_2\,|\,q)$ is defined as follows:

$$Q_n(x; t_1, t_2\,|\,q) = t_1^{-n}(t_1t_2; q)_n\, {}_3\phi_2\left(\left.{q^{-n}, t_1e^{i\theta}, t_1e^{-i\theta} \atop t_1t_2, 0}\right|\, q, q\right). \tag{4.1}$$

When $|t_1|, |t_2| < 1$, they satisfy the following orthogonality relation:

$$\int_{-1}^{1} Q_m(x; t_1, t_2\,|\,q)Q_n(x; t_1, t_2\,|\,q)w(x, t_1, t_2\,|\,q)\mathrm{d}x = (q; q)_n(t_1t_2; q)_n\delta_{m,n}, \tag{4.2}$$

where the weight function is given by

$$w(x, t_1, t_2 \,|\, q) = \frac{(q, e^{2i\theta}, e^{-2i\theta}, t_1 t_2; q)_\infty}{2\pi\sqrt{1-x^2}(t_1 e^{i\theta}, t_1 e^{-i\theta}, t_2 e^{i\theta}, t_2 e^{-i\theta}; q)_\infty} \tag{4.3}$$

with $x = \cos\theta$; see [20, (15.1.1)–(15.1.2)].

By choosing $t_3 = t_4 = 0$ in Theorem 3.5, we have the following result for the Al-Salam–Chihara weight.

Theorem 4.1 *Given $a_1, a_2 \in (-1, 1)$, let $w(x, a_1, a_2 \,|\, q)$ be the weight function of the Al-Salam–Chihara polynomials defined in (4.3). Then, we have*

$$\lim_{q\to -1} \int_{-1/\lambda_q}^{1/\lambda_q} f(\lambda_q x) w(\lambda_q x, a_1, a_2 \,|\, q)\lambda_q \mathrm{d}x = \frac{x_0 + y_0}{2x_0} f(x_0) + \frac{x_0 - y_0}{2x_0} f(-x_0),$$

for any $f \in C[-1, 1]$, where

$$x_0 = \frac{\sqrt{2 + a_1^2 + a_2^2}}{2} \quad \text{and} \quad y_0 = \frac{a_1 + a_2}{2}.$$

Moreover,

$$w(\lambda_q x, a_1, a_2 \,|\, q)\mathrm{d}x \rightharpoonup \frac{x_0 + y_0}{2x_0}\delta(x - x_0) + \frac{x_0 - y_0}{2x_0}\delta(x + x_0),$$

as $q \to -1$.

4.2 The continuous q-Hermite polynomials

If we further set $t_1 = t_2 = 0$ in (4.3), it gives us the weight function for the continuous q-Hermite polynomials in (1.6). Then, the continuous q-Hermite polynomials $H_n(x \,|\, q)$ satisfy the following orthogonality relation:

$$\int_{-1}^{1} H_m(x \,|\, q) H_n(x \,|\, q) w_H(x \,|\, q) \, \mathrm{d}x = (q; q)_n \, \delta_{m,n}.$$

By using our Theorem 3.5 and choosing the parameters $a_1 = a_2 = a_3 = a_4 = 0$, one immediately sees that Deng and Yang's conjecture [17] is true. More precisely, we have the following theorem.

Theorem 4.2 *Let $w_H(x \,|\, q)$ be the weight function of the continuous q-Hermite polynomials defined in (1.6). Then, for any $f \in C[-1, 1]$, we have*

$$\lim_{q\to -1} \int_{-1/\lambda_q}^{1/\lambda_q} f(\lambda_q x) w_H(\lambda_q x \,|\, q)\lambda_q \mathrm{d}x = \frac{1}{2} f\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{2} f\left(-\frac{\sqrt{2}}{2}\right).$$

Moreover,

$$w_H(\lambda_q x \,|\, q)\mathrm{d}x \rightharpoonup \frac{1}{2}\left(\delta(x-\frac{\sqrt{2}}{2})+\delta(x+\frac{\sqrt{2}}{2})\right),$$

as $q \to -1$.

4.3 The continuous *q*-ultraspherical polynomials

Next, we consider the continuous q-ultraspherical polynomials. Their weight function is given by

$$w(x,\beta\,|\,q)=\frac{(q,e^{2i\theta},e^{-2i\theta},\beta^2;q)_\infty}{2\pi\sqrt{1-x^2}(\beta,q\beta,\beta e^{2i\theta},\beta e^{-2i\theta};q)_\infty},\tag{4.4}$$

with $x=\cos\theta$; see [20, (13.2.4)–(13.2.5)]. When $|\beta|<1$, the continuous q-ultraspherical polynomials $C_n(x;\beta\,|\,q)$ satisfy the following orthogonality relation:

$$\int_{-1}^{1} C_m(x;\beta\,|\,q)C_n(x;\beta\,|\,q)w(x,\beta\,|\,q)\mathrm{d}x=\frac{(\beta^2;q)_n\,(1-\beta)}{(q;q)_n\,(1-\beta q^n)}\delta_{m,n}.\tag{4.5}$$

Comparing with the weight function for the Askey–Wilson polynomials in (3.3), one obtains the weight function (4.4) by choosing

$$t_1=\sqrt{\beta},\quad t_2=-\sqrt{\beta},\quad t_3=\sqrt{q\beta}\quad\text{and}\quad t_4=-\sqrt{q\beta}$$

in (3.3). One may expect analogous results also hold when we take the above special parameters in Theorem 3.5. Indeed, figures similar to those of the q-Hermite weight $w_H(x\,|\,q)$ in (1.6) can be plotted; see Fig. 1. One may compare the graphs below with those in [17, Fig. 1] and [27, Fig. 1].

Note that the above observation only holds formally. An independent proof is needed because some of the parameters t_1, t_2, t_3, t_4 may be imaginary when β or q becomes negative. Then, we have the following result.

Theorem 4.3 *Given* $\beta\in(-1,1)$*, let* $w(x,\beta\,|\,q)$ *be the weight function of the continuous* q*-ultraspherical polynomials defined in* (4.4)*. Then, we have*

$$\lim_{q\to-1}\int_{-1/\lambda_q}^{1/\lambda_q} f(\lambda_q x)w(\lambda_q x,\beta\,|\,q)\lambda_q\mathrm{d}x=\frac{1}{2}f\left(\frac{\sqrt{2}}{2}\right)+\frac{1}{2}f\left(-\frac{\sqrt{2}}{2}\right),$$

for any $f\in C[-1,1]$*. Moreover,*

$$w(\lambda_q x,\beta\,|\,q)\mathrm{d}x \rightharpoonup \frac{1}{2}\left(\delta(x-\frac{\sqrt{2}}{2})+\delta(x+\frac{\sqrt{2}}{2})\right),$$

as $q \to -1$.

Proof Similar to (3.4), let us first recall the following connection relation for the q-ultraspherical polynomials:

$$C_n(x;\gamma\,|\,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\beta^k(\gamma/\beta;q)_k(\gamma;q)_{n-k}(1-\beta q^{n-2k})}{(q;q)_k(q\beta;q)_{n-k}(1-\beta)} C_{n-2k}(x;\beta\,|\,q);$$

see [20, (13.3.1)]. Note that $U_n(x) = C_n(x;q\,|\,q)$ (cf. [20, (13.2.14)]). Then, we have from the above formula by letting $\gamma = q$:

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\beta^k(q/\beta;q)_k(q;q)_{n-k}(1-\beta q^{n-2k})}{(q;q)_k(q\beta;q)_{n-k}(1-\beta)} C_{n-2k}(x;\beta\,|\,q). \tag{4.6}$$

By the orthogonality (4.4) and $C_0(x;\beta\,|\,q) = 1$, we have

$$\int_{-1/\lambda_q}^{1/\lambda_q} U_n(\lambda_q x) w(\lambda_q x, \beta\,|\,q)\lambda_q \mathrm{d}x = \begin{cases} 0, & n = 2m+1, \\ \beta^m (q/\beta;q)_m/(q\beta;q)_m, & n = 2m. \end{cases} \tag{4.7}$$

When $n = 2m$, we have

$$\lim_{q\to -1} \int_{-1/\lambda_q}^{1/\lambda_q} U_n(\lambda_q x) w(\lambda_q x, \beta\,|\,q)\lambda_q \mathrm{d}x = (-1)^{\lfloor m/2 \rfloor} = U_n(\sqrt{2}/2).$$

When $n = 2m+1$, we have

$$\lim_{q\to -1} \int_{-1/\lambda_q}^{1/\lambda_q} U_n(\lambda_q x) w(\lambda_q x, \beta\,|\,q)\lambda_q \mathrm{d}x = 0.$$

Coupling the above two equations gives

$$\lim_{q\to -1} \int_{-1/\lambda_q}^{1/\lambda_q} U_n(\lambda_q x) w(\lambda_q x, \beta\,|\,q)\lambda_q \mathrm{d}x = \frac{1}{2} U_n\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{2} U_n\left(-\frac{\sqrt{2}}{2}\right).$$

The remaining proof of Theorem 4.3 follows from a similar argument as in the proof of Theorem 3.5, where the Weierstrass approximation theorem is used. ◻

Remark 4.4 The above theorem suggests us Theorem 3.5 remain valid when the parameters a_1, a_2, a_3, a_4 occur in pairs of complex conjugates. Without loss of generality, we assume that either $a_1 = \bar{a}_2$ or both a_1 and a_2 are real, and either $a_3 = \bar{a}_4$ or both a_3 and a_4 are real. We further assume that $|a_1| < 1,\ |a_2| < 1,\ |a_3| < 1,\ |a_4| < 1$. It suffices to prove that x_0 defined in (3.17) is positive and y_0 defined in (3.20) is real.

To achieve this, we rewrite $1 - A - B = C/D$, where $D := 4(1 - a_1^2 a_2^2 a_3^2 a_4^2) > 0$, and

$$\begin{aligned} C &:= 2 - 2a_1^2 a_2^2 a_3^2 a_4^2 + \sum_{j=1}^{4} a_j^2 - \sum_{1 \le i < j < k \le 4} a_i^2 a_j^2 a_k^2 \\ &= (1 + a_1^2)(1 + a_2^2)(1 - a_3^2 a_4^2) + (1 + a_3^2)(1 + a_4^2)(1 - a_1^2 a_2^2) > 0. \end{aligned}$$

This proves that $x_0 = \sqrt{C/D} > 0$. It is obvious from (3.20) that $y_0 \in \mathbb{R}$.

5 The complex case

In this section, we consider the case when $q \to \omega_k$, where $\omega_k = e^{2\pi i/k}$ with $k = 3, 4, \ldots$. Again, we assume the recurrence coefficients $\beta_n(\omega_k)$ vanish for some finite n, which differs from the sieved polynomials considered in [2]. Let us take the continuous q-ultraspherical polynomials $C_n(x; \beta \,|\, q)$ as an example. Note that (4.6) and (4.7) still hold when q is complex. Then, we obtain

$$\lim_{q \to \omega_k} \int_{-1}^{1} U_{2m}(x) w(x, \beta \,|\, q) \mathrm{d}x = \beta^m (\omega_k/\beta; \omega_k)_m / (\omega_k \beta; \omega_k)_m$$

and

$$\lim_{q \to \omega_k} \int_{-1}^{1} U_{2m+1}(x) w(x, \beta \,|\, q) \mathrm{d}x = 0.$$

The above formulas imply

$$\begin{aligned} \lim_{q \to \omega_3} \int_{-1}^{1} U_n(x) w(x, \beta \,|\, q) \mathrm{d}x = & \frac{(\beta + 1)((\sqrt{3} + 3i)\beta + 2\sqrt{3})}{6(\sqrt{3} + i)(\beta^2 + \beta + 1)} U_n\left(\frac{\sqrt{3}}{2}\right) \\ & + \frac{2\sqrt{3}\beta^2 + \sqrt{3} + 3i}{3(\sqrt{3} + i)(\beta^2 + \beta + 1)} U_n(0) \\ & + \frac{(\beta + 1)((\sqrt{3} + 3i)\beta + 2\sqrt{3})}{6(\sqrt{3} + i)(\beta^2 + \beta + 1)} U_n\left(-\frac{\sqrt{3}}{2}\right). \end{aligned}$$

It is interesting to note that the mass points are located at 0 and $\pm\frac{\sqrt{3}}{2}$, which are independent of the parameter β. The mass points are indeed

$$\cos\left(\frac{(2j-1)\pi}{2k}\right), \qquad j = 1, \ldots, k.$$

We may continue to obtain

$$\lim_{q\to\omega_4}\int_{-1}^{1}U_n(x)w(x,\beta\,|\,q)\mathrm{d}x = C_1(\beta)\left[U_n\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)+U_n\left(-\frac{\sqrt{2-\sqrt{2}}}{2}\right)\right] +C_2(\beta)\left[U_n\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)+U_n\left(-\frac{\sqrt{2+\sqrt{2}}}{2}\right)\right]$$

for certain coefficients $C_{1,2}(\beta)$.

The monic continuous q-ultraspherical polynomials are given as follows:

$$p_n(x;\beta|q)=\frac{(q;q)_n}{2^n(\beta;q)_n}C_n(x;\beta|q)=\frac{e^{in\theta}}{2^n}\sum_{k=0}^{n}\frac{(q^{-n},\beta;q)_k}{(q,q^{1-n}/\beta;q)_k}(qe^{-2i\theta}/\beta)^k,$$

where $x=\cos\theta$. Let $n>1$. As q approaches $\omega_n=e^{2\pi i/n}$ from the interior of the unit disk, we have $q^n\to 1$ and

$$\frac{(q^{-n},\beta;q)_k}{(q,q^{1-n}/\beta;q)_k}\to 0$$

for all $1<k<n$. Thus, only the first and the last terms in the summation remain. It then follows that

$$p_n(x;\beta|\omega_n)=\frac{e^{in\theta}}{2^n}(1+e^{-2in\theta})=2^{1-n}\cos(n\theta)=2^{1-n}T_n(x),$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind. In particular, the zeros of $p_n(x;\beta|\omega_n)$ are

$$x_r=\cos\left(\frac{r-1/2}{n}\pi\right),\quad r=1,\ldots,n.$$

Note that we have fixed β and then let $q\to\omega_n$, while in [2], it was assumed that $\beta=(q/\omega_n)^{\lambda n}$ and before taking the limit $q\to\omega_n$. The difference equation for $p_n(x)=p_n(x;\beta|q)$ is

$$xp_n(x)=p_{n+1}(x)+\frac{(1-q^n)(1-\beta^2q^{n-1})}{4(1-\beta q^{n-1})(1-\beta q^n)}p_{n-1}(x).$$

Let $p_n^*(x)=p_n^*(x;\beta|q)$ be another linearly independent solution of the above equation with initial values $p_0^*(x)=0$ and $p_1^*(x)=1$. According to the argument in Sect. 1, the q-ultraspherical measure approaches the discrete measure

$$\sum_{r=1}^{n}\frac{p_n^*(x_r;\beta|\omega_n)}{p_n'(x_r;\beta|\omega_n)}\delta_{x_r},$$

as $q \to \omega_n = e^{2\pi i/n}$. Recall that $p_n(x;\beta|\omega_n) = 2^{1-n}T_n(x)$. We have $p_n'(x;\beta|\omega_n) = 2^{1-n}n\sin(n\theta)/\sin(\theta)$, and particularly,

$$p_n'(x_r;\beta|\omega_n) = \frac{2^{1-n}n\sin((r-1/2)\pi)}{\sqrt{1-x_r^2}} = \frac{(-1)^{r-1}n}{2^{n-1}\sqrt{1-x_r^2}}.$$

When $\beta = 0$, the q-ultraspherical measure reduces to the q-Hermite measure. We will show later in Lemma 5.1 that $p_n^*(x;0|\omega_n) = n(e^{i\theta}/2)^{n-1}S_{n-1}(-\omega_n e^{-2i\theta}|\omega_n)$, where S_{n-1} is the Stieltjes-Wigert polynomial of degree $n-1$. Consequently, the limit of q-Hermite measure as $q \to \omega_n$ is a discrete measure with mass points $x_r = \cos[\pi(r-1/2)/n]$ for $r = 1, \ldots, n$, and the weight at x_r is

$$\begin{aligned}\frac{p_n^*(x_r;0|\omega_n)}{p_n'(x_r;0|\omega_n)} &= (-1)^{r-1}\sin[\pi(r-1/2)/n]e^{i\pi(r-1/2)(n-1)/n}S_{n-1}(-\omega_n^{3/2-r}|\omega_n)\\ &= \frac{1-e^{-i\pi(2r-1)/n}}{2}S_{n-1}(-\omega_n^{3/2-r}|\omega_n).\end{aligned}$$

Lemma 5.1 *Let $p_n^*(x) = p_n^*(x;0|q)$ be the solution of the difference equation*

$$xp_n^*(x) = p_{n+1}^*(x) + \frac{1-q^n}{4}p_{n-1}^*(x), \quad n \ge 1,$$

with initial conditions $p_0^*(x) = 0$ and $p_1^*(x) = 1$. For any $n \ge 1$, we have

$$p_n^*(x;0|q) = \frac{(q;q)_n}{2^{n-1}}\sum_{k=1}^{\infty} q^{k-1}C_{n-1}(x;q^k|q). \tag{5.1}$$

Especially, when $q = \omega_n = e^{2i\pi/n}$, we obtain

$$p_n^*(\cos\theta;0|\omega_n) = \frac{ne^{i\theta(n-1)}}{2^{n-1}}S_{n-1}(-\omega_n e^{-2i\theta};\omega_n), \tag{5.2}$$

where S_{n-1} is the Stieltjes-Wigert polynomial of degree $n-1$.

Proof Define the generating function

$$F(x,t) = \sum_{n=1}^{\infty}\frac{p_n^*(x)t^n}{(q;q)_n} = \frac{t}{1-q} + \sum_{n=2}^{\infty}\frac{p_n^*(x)t^n}{(q;q)_n}.$$

We have from the difference equation that

$$\sum_{n=1}^{\infty}\frac{xp_n^*(x)t^n}{(q;q)_n} = \frac{1}{t}\sum_{n=1}^{\infty}\frac{p_{n+1}^*(x)t^{n+1}(1-q^{n+1})}{(q;q)_{n+1}} + \frac{t}{4}\sum_{n=1}^{\infty}\frac{p_{n-1}^*(x)t^{n-1}}{(q;q)_{n-1}},$$

which gives the functional equation for the generating function

$$xF(x,t) = \frac{1}{t}[F(x,t) - F(x,tq)] - 1 + \frac{t}{4}F(x,t).$$

The above equation can be rewritten as follows:

$$(1 - xt + t^2/4)F(x,t) = F(x,tq) + t.$$

Set $x = \cos\theta$ and

$$F(x,t) = \frac{G(x,t)}{(te^{i\theta}/2, te^{-i\theta}/2; q)_\infty}.$$

It follows that

$$G(x,t) = G(x,tq) + t(tqe^{i\theta}/2, tqe^{-i\theta}/2; q)_\infty.$$

Iterating the above equation and making use of $G(x,0) = 0$, we have

$$G(x,t) = \sum_{k=1}^{\infty} tq^{k-1}(tq^k e^{i\theta}/2, tq^k e^{-i\theta}/2; q)_\infty,$$

which yields

$$F(x,t) = \sum_{k=1}^{\infty} \frac{tq^{k-1}}{(te^{i\theta}/2, te^{-i\theta}/2; q)_k}.$$

Because of the q-binomial theorem, we can expand

$$F(x,t) = t\sum_{k=1}^{\infty} q^{k-1} \sum_{j=0}^{\infty} \frac{(q^k;q)_j}{(q;q)_j}(te^{i\theta}/2)^j \sum_{l=0}^{\infty} \frac{(q^k;q)_l}{(q;q)_l}(te^{-i\theta}/2)^l.$$

Consequently,

$$\begin{aligned} p_n^*(x) &= \frac{(q;q)_n}{2^{n-1}} \sum_{k=1}^{\infty} q^{k-1} \sum_{j=0}^{n-1} \frac{(q^k;q)_j (q^k;q)_{n-1-j}}{(q;q)_j (q;q)_{n-1-j}} e^{i\theta(n-1-2j)} \\ &= \frac{(q;q)_n}{2^{n-1}} \sum_{k=1}^{\infty} q^{k-1} C_{n-1}(x; q^k|q), \quad n \ge 1. \end{aligned}$$

This proves (5.1). Now, let us consider the case when $q \to \omega_n = e^{2i\pi/n}$. By q-binomial theorem, we have

$$
\begin{aligned}
&(q;q)_n \sum_{k=1}^{\infty} q^{k-1}(q^k;q)_r(q^k;q)_s \\
&\quad = (q;q)_n \sum_{k=1}^{\infty} q^{k-1} \sum_{l=0}^{r} \frac{(q^{-r};q)_l}{(q;q)_l} q^{l(k+r)} \sum_{m=0}^{s} \frac{(q^{-s};q)_m}{(q;q)_m} q^{m(k+s)} \\
&\quad = \sum_{l=0}^{r}\sum_{m=0}^{s} \frac{(q^{-r};q)_l(q^{-s};q)_m}{(q;q)_l(q;q)_m} q^{l(1+r)+m(1+s)} \frac{(q;q)_n}{1-q^{1+l+m}}.
\end{aligned}
$$

Assume $r+s=n-1$. Each term, except for $l=r$ and $m=s$, in the above double sum vanishes as $q \to \omega_n$. Consequently, the limit of the above double sum is

$$
\begin{aligned}
&\lim_{q\to\omega_n} \frac{(q^{-r};q)_r(q^{-s};q)_s}{(q;q)_r(q;q)_s} q^{r(1+r)+s(1+s)}(q;q)_{n-1} \\
&\quad = \lim_{q\to\omega_n} (-1)^{r+s} q^{-r(r+1)/2-s(s+1)/2+r(1+r)+s(1+s)}(q;q)_{n-1} \\
&\quad = (-1)^{n-1} n w_n^{(r^2+r+s^2+s)/2},
\end{aligned}
$$

where in the last step, we have made use of the equality $(w_n;w_n)_{n-1}=n$. Note that w_n^k with $k=1,\ldots,n$ are the distinct roots of $z^n-1=0$. We have

$$
f(z) = \prod_{k=1}^{n-1}(z-w_n^k) = \frac{z^n-1}{z-1} = \sum_{k=0}^{n-1} z^k.
$$

Especially, $(w_n;w_n)_{n-1} = f(1) = n$. Summarizing the above arguments, we obtain

$$
\begin{aligned}
\lim_{q\to\omega_n} p_n^*(x) &= \frac{(-1)^{n-1}n}{2^{n-1}} \sum_{j=0}^{n-1} \frac{w_n^{(j^2+j+(n-1-j)^2+(n-1-j))/2}}{(w_n;w_n)_j(w_n;w_n)_{n-1-j}} e^{i\theta(n-1-2j)} \\
&= \frac{(-1)^{n-1}n}{2^{n-1}} \omega_n^{n(n-1)/2} e^{i\theta(n-1)} S_{n-1}(-\omega_n^{-n+1}e^{-2i\theta};\omega_n), \\
&= \frac{ne^{i\theta(n-1)}}{2^{n-1}} S_{n-1}(-\omega_n e^{-2i\theta};\omega_n),
\end{aligned}
$$

which proves (5.2). □

6 Conclusion and discussion

In this paper, we have used two different approaches to derive the weak limit of the Askey–Wilson measure as q tends to -1 from the right. We proved that the limiting

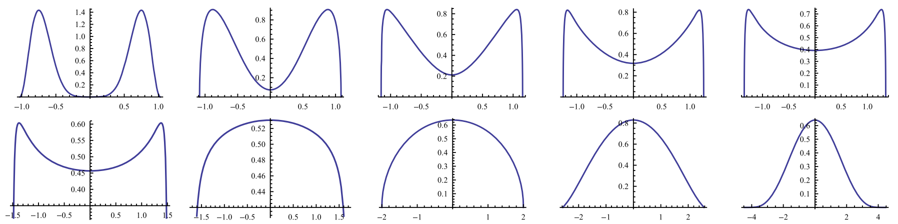

Fig. 1 Plots of the q-ultraspherical measure $w(\lambda_q x, \beta \mid q)$ in (4.4) with $\lambda_q = \sqrt{1-q}$ and $\beta = 0.5$. In the first row, q is negative and increases from -0.9 to -0.1 with a step size 0.2. While in the second row, q is positive and increases from 0.1 to 0.9 with a step size 0.2

measure is discrete, and it has exactly two mass points that are symmetric about the origin. As special cases, our results can be applied to the Al-Salam-Chihara polynomials, the continuous q-Hermite polynomials, and the continuous q-ultraspherical polynomials. Especially, we proved a conjecture made by Deng and Yang [17] who observed from the numerical graph that the weak limit of q-Hermite measure as $q \to -1$ is the sum of two discrete masses with equal weight at $\pm\sqrt{2}/2$. For the complex case when q approaches a root of unity $\omega_n = e^{2\pi i n}$ with $n = 3, 4, \ldots$, we also calculated the weak limit of the q-ultraspherical measure which is a discrete measure with n mass points $\cos[(r-1/2)\pi/n]$ for $r = 1, 2, \ldots, n$. For the special case of q-Hermite polynomials, we also expressed the weights explicitly in terms of the Stieltjes-Wigert polynomials.

Another interesting phenomenon observed from the numerical simulation is that the shape of the orthogonality measure for the q-orthogonal polynomials changes from unimodal to bimodal as q decreases from 1 to -1. Taken the q-ultraspherical measure in (4.4) as an example, we observe from Fig. 1 that there exists a critical value $q^* \in (0.1, 0.2)$ such that the measure is unimodal when $q \in (q^*, 1)$ and bimodel when $q \in (-1, q^*)$. Here, we have chosen $\beta = 0.5$. It is worth investigating how the critical value q^* depends on the parameter β.

There are cases where the moments do no not determine the measure uniquely. These moment problems are called indeterminate; see [1, 25]. The best-known example is the q^{-1}-Hermite polynomials of Askey [5] and Ismail and Masson [21]. We know many measures for the q^{-1}-Hermite polynomials including the N-extremal measures which make the polynomials complete in their weighted L^2 spaces. The coefficient of p_{n-1} in the monic form of the recurrence relation contains the factor q^{-n} which becomes negative as soon as q turns negative. This makes the limit as $q \to -1$ different from those studied in the present paper. In particular, the N-extremal measures will not converge to a probability measure as $q \to -1$. However, we expect the limit may still contain two mass points where the signed weights add to one. We will explore this in future work.

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